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Spectral theory of discrete processes

Research Article

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Abstract: We offer a spectral analysis for a class of transfer operators. These transfer operators arise for a wide range of stochastic processes, ranging from random walks on infinite graphs to the processes that govern signals and recursive wavelet algorithms; even spectral theory for fractal measures. In each case, there is an associated class of harmonic functions which we study. And in addition, we study three questions in depth:

In specific applications, and for a specific stochastic process, how do we realize the transfer operator T as an operator in a suitable Hilbert space? And how to spectral analyze T once the right Hilbert space $\mathcal{H}$ has been selected? Finally we characterize the stochastic processes that are governed by a single transfer operator.

In our applications, the particular stochastic process will live on an infinite path-space which is realized in turn on a state space S. In the case of random walk on graphs G, S will be the set of vertices of G. The Hilbert space $\mathcal{H}$ on which the transfer operator T acts will then be an L^2 space on S, or a Hilbert space defined from an energy-quadratic form.

This circle of problems is both interesting and non-trivial as it turns out that T may often be an unbounded linear operator in $\mathcal{H}$; but even if it is bounded, it is a non-normal operator, so its spectral theory is not amenable to an analysis with the use of von Neumann's spectral theorem. While we offer a number of applications, we believe that our spectral analysis will have intrinsic interest for the theory of operators in Hilbert space.

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1. Introduction

In this paper, we consider infinite configurations of vectors $(f_k)_{k\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$. Since our Hilbert spaces $\mathcal{H}$ are typically infinite-dimensional, this can be quite complicated, and it will be difficult to make sense of finite and infinite linear combinations $\sum_{k\in\mathbb{Z}} c_k f_k$.

In case the system (f_k) is orthogonal, the problem is easy,

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but non-orthogonality serves as an encoding of statistical correlations, which in turn motivates our study. In applications, a particular system of vectors f_k may often be analyzed with the use of a single unitary operator U in $\mathcal{H}$. This happens if there is a fixed vector $\varphi \in \mathcal{H}$ such that $f_k = U^k\varphi$ for all $k \in \mathbb{Z}$. When this is possible, the spectral theorem will then apply to this unitary operator. A key idea in our paper is to identify a spectral density function and a transfer operator, both computed directly from the pair (φ, U).

We show that the study of linear expressions $\sum_k c_k f_k$ may be done with the aid of the spectral function for a pair (φ, U). A spectral function for a unitary operator U is really a system of functions (p_φ), one for each cyclic subspace $\mathcal{H}(\varphi)$. In each cyclic subspace, the function p_φ is a complete unitary invariant for U restricted to $\mathcal{H}(\varphi)$: by this we mean that the function p_φ encodes all the spectral data coming from the vectors $f_k = U^k\varphi$, $k \in \mathbb{Z}$. For background literature on spectral function and their applications we refer to [1, 10, 16, 19–21].

In summary, the spectral representation theorem is the assertion that commuting unitary operators in Hilbert space may be represented as multiplication operators in an L^2-Hilbert space. The understanding is that this representation is defined as a unitary equivalence, and that the L^2-Hilbert space to be used allows arbitrary measures, and L^2 will be a Hilbert space of vector valued functions, see e.g., [6]. Because of applications, our systems of vectors will be indexed by an arbitrary discrete set rather than merely integers $\mathbb{Z}$.

We will attack this problem *via* an isometric embedding of $\mathcal{H}$ into an L^2-space built on infinite paths in such a way that the vectors f_k in $\mathcal{H}$ transform into a system of random variables Z_k. Specifically, *via* certain encodings we build a path-space Ω for the particular problem at hand as well as a path space measure $\mathbb{P}$ defined on a σ-algebra of subsets of Ω.

If $\mathcal{H}$ consists of a space of functions f on a state space S, we will need the covariance numbers

$$\mathbb{E}((f_1 \circ Z_n) \cdot (f_2 \circ Z_m)) \equiv \int_\Omega f_1(Z_n(\gamma))f_2(Z_m(\gamma))d\mathbb{P}(\gamma),$$

where $Z_n : \Omega \to S$, *i.e.*, where the stochastic process is S-valued. The set S is called the state space.

The paper is organized as follows. In Sec. 2, for later use, we present our path-space approach, and we discuss the path-space measures that we will use in computing transitions for stochastic processes. We prove two theorems making the connection between our path-space measures on the one hand, and the operator theory on the other. Several preliminary results are established proving how the transfer operator governs the process and its applications.

The applications we give in Sec. 3 and 4 are related. In fact, we unify these applications with the use of an encoding map which is also studied in detail. It is applied to transitions on certain infinite graphs, to dynamics of (non-invertible) endomorphisms (measures on solenoids), to digital filters and their use in wavelets and signals, and to harmonic analysis on fractals.

The remaining sections deal primarily with applications to a sample of concrete cases.

2. Stochastic processes

A key tool in our analysis is the construction of path-space measures on infinite paths, primarily in the case of discrete paths, but the fundamental ideas are the same in the continuous case. Both viewpoints are used in [12]. Readers who wish to review the ideas behind there constructions (stochastic processes and consistent families of measures) are referred to [7–9] and [18].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Borel probablity space, Ω compact Hausdorff space. (Expectation $\mathbb{E}(\cdot) = \int_\Omega \cdot d\mathbb{P}$.)

Let $(Z_k)_{k\geq 0}$ be a stochastic process, and

$$\mathcal{F}_n = \sigma\text{-alg.}\{Z_k | k \leq n\}, \tag{1}$$

the corresponding filtration. Let $\mathcal{A}_n \equiv$ the subspace in $L^2(\Omega, \mathbb{P})$ generated by $\mathcal{F}_n$. Let P_n be the orthogonal projection of $L^2(\Omega, \mathbb{P})$ onto $\mathcal{A}_n$; then the conditional expectations $\mathbb{E}(\cdot|\mathcal{F}_n)$ is simply $= P_n$.

We say that $(Z_k)_{k\geq 0}$ has the generalized Markov property if and only if there exists a state space S (also a compact Borel space):

$$Z_k : \Omega \to S,$$

such that for all bounded functions f on S, for all $n \in \mathbb{N}_{\geq 0}$, $\mathbb{E}(f|\mathcal{F}_n) = \mathbb{E}(f|Z_n)$.

To make precise the operator theoretic tools going into our construction, we must first introduce the ambient Hilbert spaces. We are restricting here to L^2 processes, so the corresponding stochastic integrals will take values in an ambient L^2-space of random variables: For our analysis, we must therefore specify a fixed probability space, with σ-algebra and probability measure.

We will have occasion to vary this initial probability space, depending on the particular transition operator that governs the process.

In the most familiar case of Brownian motion, or random walk, the probability space amounts to a somewhat standard construction of Wiener and Kolmogorov, but here with

some modification for our problem at hand: The essential axiom in Wiener's case is that all finite samples are jointly Gaussian, but we will drop this restriction and consider general stochastic processes, and so we will not make restricting assumptions on the sample distributions and on the underlying probability space. For more details, and concrete applications, regarding this stochastic approach and its applications, see Sec. 2 and 4 below.
We begin here with a particular case of a process taking values in the set of vertices in a fixed infinite graph G: [13]

2.1. Starting assumptions and constructions

(a) $G = (G^0, G^1)$ a graph, G^0 = the set of vertices, G^1 = the set of edges.

(b) $(S, \mathcal{B}_S, \mu)$ a probability space.

(c) The *transition matrix* is the function

$$p(x,y) \equiv \mathbb{P}(\{\gamma \in \Omega | Z_n(\gamma) = x, Z_{n+1}(\gamma) = y\})$$

defined for all $(x, y) \in G^1$, and we assume that it is independent of n.

(d) From (a) and (b), we construct the path space

$$\Omega \equiv \{\gamma = (x_0 x_1 x_2 \cdots) | (x_{i-1} x_i) \in G^1, \forall i \in \mathbb{N}\},$$

and the path-measure $\mathbb{P} = \mathbb{P}_\mu$. The cylinder sets given by the following data: For $E_i \in \mathcal{B}_S$, $x_i \subset S$, set

$$\mathbb{P}(C(E_1, \cdots, E_n)) \equiv \int_{E_0}\int_{E_1}\cdots\int_{E_n} p(x_0,x_1)p(x_1,x_2) \cdots p(x_{n-1},x_n)d\mu(x_0)d\mu(x_1)\cdots d\mu(x_n).$$

(e) Starting with $(\Omega, \mathcal{F}, \mathbb{P})$, if $\mathcal{G} \subset \mathcal{F}$ is a subsigma algebra, let $\mathbb{E}(\cdot|\mathcal{G})$ be the conditional expectation, conditioned by $\mathcal{G}$.
If (X_i) is a family of random variables, and $\mathcal{G}$ is the σ-algebra generated by (X_i) we write $\mathbb{E}(\cdot|(X_i))$ in place of $\mathbb{E}(\cdot|\mathcal{G})$.

(f) Let $(\Omega, \mathcal{F}, \mathbb{P}, (Z_n))$ be as above. We say that (Z_n) is *Markov* if and only if

$$\mathbb{E}(f \circ Z_{n+1} | \{Z_0, \cdots Z_n\}) = \mathbb{E}(f \circ Z_{n+1}|Z_n) \quad \text{for all } n \in \mathbb{N}_0.$$

(g) From (b) and (d) we define the *transfer operator* T by

$$(Tf)(x) = \int_S p(x,y)f(y)d\mu(y) \tag{2}$$

for measurable functions f on S. If $\mathbb{1}$ denote the constant function 1 on S, then $T\mathbb{1} = \mathbb{1}$.

(h) Let $(S, \mathcal{B}_S, \mu)$ and T be as in (g), see(2). A measure μ_0 on S is said to be a *Perron-Frobenius measure* if and only if

$$\int_S (Tf)(x)d\mu_0(x) = \int_S f(x)d\mu_0(x), \quad \text{abbreviated } \mu_0 \circ T = \mu_0. \tag{3}$$

(i) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be as above, and let T be the transfer operator. If μ_0 is a Perron-Frobenius measure, let $\mathbb{P}^{(\mu_0)}$ be the measure on Ω determined by using μ_0 as the first factor, *i.e.*,

$$\begin{aligned}\mathbb{P}^{(\mu_0)}(C(E_1,\cdots,E_n)) &= \int_{E_0}\int_{E_1}\cdots\int_{E_n} p(x_0,x_1)p(x_1,x_2) \cdots p(x_{n-1},x_n)d\mu_0(x_0)d\mu(x_1)\cdots d\mu(x_n)\\ &= \int_{E_0}\mathbb{P}_{x_0}(C(E_1,\cdots,E_n))d\mu(x_0).\end{aligned}$$

In many cases, it is possible to choose specific Perron-Frobenius measures μ_0, *i.e.*, measures μ_0 satisfying

$$\mu_0(S) = 1 \text{ and } \int_S (Tf)(x)d\mu_0(x) = \int_S f(x)d\mu_0(x).$$

(Note the normalization!)

Theorem 2.1.
(D. Ruelle) [2] Suppose there is a norm $\|\cdot\|$ on bounded measurable functions f on S such that the $\|\cdot\|$-completion $L(S)$ is embedded in $L^\infty(S)$, and that there are constants $\alpha \in (0,1)$, $M \in \mathbb{R}_+$ such that

$$\|Tf\| \le \alpha\|f\| + M\|f\|_\infty,$$

where $\|\cdot\|_\infty$ is the essential supremum-norm. Then T has a Perron-Frobenius measure.

Theorem 2.2.
Let (S, μ) be a probability space with S carrying a separate σ-algebra $\mathcal{B}_S$ and μ defined on $\mathcal{B}_S$. Let Ω be the path space, and suppose the transfer operator T has a Perron-Frobenius measure μ_0, then

$$\mathbb{E}^{(\mu_0)}((\overline{\varphi} \circ Z_n)(\psi \circ Z_{n+1})) = \langle \varphi, T\psi \rangle_{L^2(\mu_0)} \tag{4}$$

for all $\varphi, \psi \in L^2(\mu)$, and all $n \in \mathbb{N}_0$. Here $\mathbb{E}(F) \equiv \int_\Omega F(\omega) d\mathbb{P}(\omega)$ for all integrable random variables $F : \Omega \to \mathbb{C}$; $\mathbb{E}$ for expectation.

Proof.

$$\begin{aligned}\mathbb{E}^{(\mu_0)}((\overline{\varphi} \circ Z_n)(\psi \circ Z_{n+1})) &= \int_S \int_S \cdots \int_S p(x_0, x_1) \cdots p(x_{n-1}, x_n) p(x_n, x_{n+1}) \overline{\varphi}(x_n) \psi(x_{n+1}) d\mu_0(x_0) d\mu(x_1) \cdots d\mu(x_{n+1}) \\ &= \int_S \int_S \cdots \int_S p(x_0, x_1) \cdots p(x_{n-1}, x_n) \overline{\varphi}(x_n) (T\psi)(x_n) d\mu_0(x_0) d\mu(x_1) \cdots d\mu(x_n) \\ &= \int_S T^n(\overline{\varphi} \cdot (T\psi))(x_0) d\mu_0(x_0) \\ &= \int_S \overline{\varphi}(x)(T\psi)(x) d\mu_0(x) \quad \text{by Perron-Frobenius} \\ &= \langle \varphi, T\psi \rangle_{L^2(\mu_0)}.\end{aligned}$$

□

It is not necessary in (4) to restrict attention to functions φ, ψ in $L^2(\mu_0)$. The important thing is that the integral $\int_S \overline{\varphi(x)}(T\psi)(x) d\mu_0(x)$ exists, and this quantity may then be used instead on the RHS in (4).
Let $(Z_n)_{n \in \mathbb{N}_0}$ be a stochastic process, and let $\mathcal{F}_n$ be the σ-algebra generated by $\{Z_k |\ 0 \leq k \leq n\}$. Futhermore, let $\mathbb{E}(\cdot|\mathcal{F}_n)$ be the conditioned expectation conditioned by $\mathcal{F}_n$.

Theorem 2.3.
Let $(Z_n)_{n \in \mathbb{N}_0}$ be a stochastic process with stationary transitions and operator T. Then

$$\mathbb{E}(f \circ Z_{n+1} | \mathcal{F}_n) = (Tf) \circ Z_n \tag{5}$$

for all bounded measurable functions f on S, and all $n \in \mathbb{N}_0$.

Proof. We may assume that f is a real valued function on S. Let $\mathcal{A}_n \equiv$ all bounded $\mathcal{F}_n$-measurable functions. Then the assertion in (5) may be restated as:

$$\int_\Omega \varphi(f \circ Z_{n+1}) d\mathbb{P} = \int_\Omega \varphi((Tf) \circ Z_n) d\mathbb{P} \tag{6}$$

for all $\varphi \in \mathcal{A}_n$.
If $\varphi \in \mathcal{A}_n$, $\varphi(\cdot) = \Phi(x_0, x_1, \cdots x_n)$; and then the LHS in (6) may be written as

$$\begin{aligned}&\int_S \int_S \cdots \int_S p(x_0, x_1) \cdots p(x_n, x_{n+1}) \Phi(x_0, x_1 \cdots x_n) f(x_{n+1}) d\mu_0(x_0) d\mu(x_1) \cdots d\mu(x_{n+1}) \\ &= \int_S \int_S \cdots \int_S p(x_0, x_1) \cdots p(x_{n-1}, x_n) \Phi(x_0, x_1 \cdots x_n)(Tf)(x_n) d\mu_0(x_0) d\mu(x_1) \cdots d\mu(x_n) \\ &= \int_\Omega \varphi \cdot (Tf) \circ Z_n \ d\mathbb{P}.\end{aligned}$$

Hence (5) follows. □

Corollary 2.1.
Let $(\Omega, \mathcal{F}, \mathbb{P}, (Z_n))$ be as in the theorem. Then the process (Z_n) is Markov.

Proof. We must show that

$$\mathbb{E}(f \circ Z_{n+1} | \mathcal{F}_n) = \mathbb{E}(f \circ Z_{n+1} | Z_n).$$

By the theorem, we only need to show that

$$\mathbb{E}(f \circ Z_{n+1} | Z_n) = (Tf) \circ Z_n.$$

In checking this we use the transition operator T. As a result we may now assume that φ has the form $\varphi = g \circ Z_n$ for g a measurable function on S. Hence

$$\begin{aligned} \int_\Omega \varphi (f \circ Z_{n+1}) d\mathbb{P} &= \int_\Omega (g \circ Z_n)(f \circ Z_{n+1}) d\mathbb{P} = \langle g, Tf \rangle_{L^2(\mu)} \\ &= \int_S g(Tf) d\mu = \int_\Omega (g \circ Z_n)((Tf) \circ Z_n) d\mathbb{P} \\ &= \int_\Omega \varphi ((Tf) \circ Z_n) d\mathbb{P}, \end{aligned}$$

which is the desired conclusion. □

Definition 2.1.
We say that a measurable function f on S is harmonic if $Tf = f$.

Definition 2.2.
A sequence of random variables (F_n) is said to be a martingale if and only if $\mathbb{E}(F_{n+1} | \mathcal{F}_n) = F_n$ for all $n \in \mathbb{N}_0$.

Corollary 2.2.
Let $(Z_n)_{n \in \mathbb{N}_0}$ be a stochastic process with stationary transitions and operator T. Let f be a measurable function on S.
Then f is harmonic if and only if $(f \circ Z_n)_{n \in \mathbb{N}_0}$ is a martingale.

Proof. This follows from (5) combined with Definition 2.2. □

Corollary 2.3.
Suppose a process $(Z_n)_{n \in \mathbb{N}_0}$ is stationary with a fixed transition operator $T : L^2(\mu) \to L^2(\mu)$. Then $\mu = \mathbb{P} \circ Z_n^{-1}$ for all $n \in \mathbb{N}_0$.

Proof. Let f and g be a pair of functions on S as specified above. Then we showed that

$$\int_S g f d\mu = \int_\Omega (g \circ Z_n)(f \circ Z_n) d\mathbb{P},$$

which is the desired conclusion. □

2.2. Martingales and boundaries

Let $G = (G^0, G^1)$ be an infinite graph with a fixed conductance c, and let the corresponding operators be Δ_c and T_c.
Let $h : G^0 \to \mathbb{R}$ is a harmonic function, *i.e.*, $\Delta_c h = 0$, or equivalently $T_c h = h$.
As an application of Corollary 2.2, we may then apply a theorem of J. Doob to the associated martingale $h \circ Z_n$, $n \in \mathbb{N}_0$. This means that the sequence $(h \circ Z_n)$ will then have $\mathbb{P}$- a. e. limit *i.e.*,

$$\lim_{n \to \infty} h \circ Z_n = v \quad \text{pointwise} \quad \mathbb{P} \text{ a.e.} \tag{7}$$

The limit function $v : \Omega \to \mathbb{R}$ will satisfy $v(x_0 x_1 x_2 \cdots) = v(x_1 x_2 x_3 \cdots)$, or equivalently,

$$v = v \circ \sigma. \tag{8}$$

The existence of the limit in (7) holds if one or the other of the two conditions is satisfied:

(i) $h \in L^\infty$; or

(ii) $\sup_n \int_\Omega |h \circ Z_n|^2 d\mathbb{P} < \infty$.

Proposition 2.1.
[11] If $h : G^0 \to \mathbb{R}$ is harmonic and if (i) or (ii) hold, then

$$h(x) = \int_\Omega v \, d\mathbb{P}_x \quad \text{for all } x \in G^0, \tag{9}$$

where $\mathbb{P}_x$ = the measure $\mathbb{P}$ conditioned with $Z_0(\gamma) = x$. The converse implication holds as well.

Proof. Starting with h harmonic, if the Doob-limit v in (7) exists, then it is clear that v satisfies (8). By Dominated Convergence, (9) will be satisfied.
Conversely, suppose some measurable $v : \Omega \to \mathbb{R}$ satisfies (8), and the integral in (9) exists then

$$\begin{aligned}(T_c h)(x) &= \sum_{y\sim x} p(x,y)h(y)\\ &= \sum_{y\sim x} \mathbb{P}(Z_0 = x, Z_1 = y)\mathbb{E}(v|Z_0(\cdot) = y)\\ &\underset{\text{by (8)}}{=} \sum_{y\sim x} p(x,y)\mathbb{E}_x(v|Z_1(\cdot) = y)\\ &= \sum_{y\sim x} p(x,y)\mathbb{E}(v|Z_0 = x, Z_1 = y)\\ &= \mathbb{P}_x(v(\cdots))\\ &= h(x),\end{aligned}$$

showing that h is harmonic. □

2.3. Solenoids

Example 2.1.
Let S be a compact Hausdorff space, and $\sigma : S \to S$ a finite-to-one endomorphism onto S. Let $X_\sigma(S)$ be the corresponding solenoid:

$$X_\sigma(S) \subset \prod_{n\in\mathbb{N}_0} S, \quad \text{where } \mathbb{N}_0 = \{0\}\cup\mathbb{N} = \{0,1,2,3,\cdots\},$$

$$X_\sigma(S) = \{(x_k)_{k\in\mathbb{N}_0} | \sigma(x_{k+1}) = x_k\}. \tag{10}$$

One advantage of a choice of solenoid over the initial endomorphism $\sigma : S \to S$ is that σ induces an *automorphism* $\widehat{\sigma} : X_\sigma(S) \to X_\sigma(S)$ as follows:

$$\widehat{\sigma}((x_0x_1x_2\cdots)) = (\sigma(x_0)x_0x_1x_2\cdots),$$

with inverse

$$\widehat{\sigma}^{-1}((x_0x_1x_2\cdots)) = (x_1x_2x_3\cdots).$$

Let $W : S \to [0,1]$ be a Borel measurable function, and set

$$(T_W f)(x) = \sum_{\substack{y\\ \sigma(y)=x}} W(y)f(y), \quad f\in B(S), x\in S. \tag{11}$$

Assume

$$\sum_{\sigma(y)=x} W(y) \equiv 1, \forall x\in S. \tag{12}$$

For points $x\in S$, set $D(x) \equiv \sharp\{y|\sigma(y) = x\}$. A measure μ on S is said to be *strongly invariant* if

$$\int_S \frac{1}{D(x)} \sum_{\substack{y\\ \sigma(y)=x}} f(y)d\mu(x) = \int_S f(x)d\mu(x).$$

Lemma 2.1.
Assume a measure μ on S is strongly invariant, and let m be a function on S. Set $Vf(x) = m(x)f(\sigma(x))$. Then the adjoint operator

$$V^* : L^2(\mu) \to L^2(\mu) \quad \textit{is} \quad (V^*f)(x) = \frac{1}{D(x)} \sum_{\substack{y\\ \sigma(y)=x}} \overline{m}(y)f(y).$$

Proof. See [11]. □

Set $\Omega \equiv X_\sigma(S)$ and equip it with the σ-algebra $\mathcal{F}$ and the topology which is generated by the cylinder sets.
Set $Z_k : \Omega \to S$,

$$Z_k(x_0x_1x_2\cdots) \equiv x_k, \quad k\in\mathbb{N}_0. \tag{13}$$

Let $E \subset S$ be a Borel set, and consider

$$Z_k^{-1}(E) = \{\omega\in\Omega | Z_k(\omega)\in E\}. \tag{14}$$

Then the σ-algebra $\mathcal{F}$ on Ω is generated by the sets

$$Z_k^{-1}(E) \quad \text{as } k \text{ and } E \text{ vary.} \tag{15}$$

Set

$$\mathcal{F}_n \equiv \sigma\text{-algebra} \ll Z_k | k\le n \gg, \tag{16}$$

where $\ll\cdot\gg$ refers to the σ-algebra as specified in (14).
In $\Omega = X_\sigma(S)$, consider the following random walk: For points $x, y\in S$, a transition $x\to y$ is possible if and only if $\sigma(y) = x$; and in this case the transition probability is $p_W(x,y) \equiv W(y)$.
Let μ be a probability measure on S. In Ω we introduce the following Kolmogorov measure $\mathbb{P}\equiv\mathbb{P}_W$ which is determined on cylinder sets as follows

$$\begin{aligned}\mathbb{P}(C_n) \equiv \mathbb{P}(C(E_0, E_1, E_2\cdots, E_n)) = \int_{E_0}\int_{E_1}\\ \cdots\int_{E_n} W(x_1)W(x_2)\cdots W(x_n)d\mu(x_0)d\mu(x_1)\cdots d\mu(x_n).\end{aligned} \tag{17}$$

More specifically, $\mathbb{P}$ is a measure on infinite paths, and

$$\begin{aligned}C_n = \{\omega = (\omega_0\omega_1\omega_2\cdots) | \sigma(\omega_{i+1}) = \omega_i,\\ Z_k(\omega)\in E_k, \text{ for } 0\le k\le n\}.\end{aligned} \tag{18}$$

Example 2.2.

The following is a solenoid which is used in both number theory (the study of algebraic irrational numbers) and in ergodic systems. [4]. For this family of examples, the solenoids are associated with specific polynomials $p \in \mathbb{Z}[x]$.
Let $S \equiv \mathbb{T}^s$ where $s \in \mathbb{N}$ is fixed; and let $p(x) = a_0x^s + a_1x^{s-1} + \cdots + a_s$; $a_0 \neq 0$, be a polynomial, $p \in \mathbb{Z}[x]$. Set

$$F = F_p \equiv \begin{pmatrix} 0 & a_0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_0 \\ -a_s & -a_{s-1} & \cdots & \cdots & -a_2 & -a_1 \end{pmatrix}.$$

Consider the shift σ on the infinite torus $\prod_{\mathbb{Z}} \mathbb{T}^s = (\mathbb{T}^s)^{\mathbb{Z}}$, and set

$$X_\sigma \equiv \{(z_n)_{n\in\mathbb{Z}} \in (\mathbb{T}^s)^{\mathbb{Z}} | a_0 z_{n+1} = F z_n\}.$$

Then it follows that $X_\sigma(p)$ is σ-invariant and closed. As a result, $X_\sigma(p)$ is a compact solenoid.

3. Graphs

One additional application of these ideas is to infinite graph systems (G, c) where G is a graph and c is a positive conductance function. A comprehensive study of this class of examples was carried out in the paper [12]. We will adapt the convention from that paper:

G^0 : the set of vertices in G;

G^1 : the set of edges in G;

and $c : G^1 \to \mathbb{R}_+$ the conductance function.

Assumptions

(i) *Edge symmetry.* If $x, y \in G^0$ and $(x, y) \in G^1$, then we assume that $c_{x,y} = c_{y,x}$. Moreover, $(x, y) \in G^1 \Leftrightarrow (y, x) \in G^1$.

(ii) *Finite neighborhoods.* For all $x \in G^0$, the set $Nbh(x) = \{y \in G^0 | (x, y) \in G^1\}$ is finite.

(iii) *No self-loops.* If $x \in G^0$, then $x \notin Nbh(x)$.
Convention: If $x, y \in G^0$, we write $x \sim y$ iff $(x, y) \in G^1$.

(iv) *Connectedness.* For all $x, y \in G^0$ there exists $\{x_i\}_{i=0}^n \subset G_0$ such that $(x_i, x_{i+1}) \in G^1$, $i = 0, 1, \cdots, n-1$ $x_0 = x$ and $x_n = y$.

(v) *Choice of origin.* We select an origin $o \in G^0$.

Definition 3.1.

- The Laplace operator $\Delta = \Delta_c$:

$$(\Delta f)(x) \equiv \sum_{y\sim x} c_{x,y}(f(x) - f(y)).$$

- Hilbert spaces:

(i) $l^2(G^0)$: functions $f : G^0 \to \mathbb{C}$ such that $\|f\|_2^2 = \sum_{x\in G^0} |f(x)|^2 < \infty$. Set $\langle f_1, f_2\rangle_2 \equiv \sum_{x\in G^0} \overline{f_1(x)} f_2(x)$. For every $x \in G^0$, set $\delta_x : G^0 \to \mathbb{R}$,

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

Note that $\{\delta_x\}$ is an orthonormal basis (ONB) in $l^2(G^0)$.

(ii) $\mathcal{H}_E$: finite energy functions module constants:

$$\|f\|_E^2 = \frac{1}{2} \sum_{\text{all}} \sum_{x\sim y} c_{x,y} |f(x) - f(y)|^2. \tag{19}$$

Set

$$\langle f_1, f_2\rangle_E \equiv \frac{1}{2} \sum \sum_{x\sim y} c_{x,y} (\overline{f_1(x)} - \overline{f_1(y)})(f_2(x) - f_2(y)). \tag{20}$$

- Dipoles. For all $x \in G^0$ there is a unique $v_x \in \mathcal{H}_E$ such that

$$\langle v_x, f\rangle_E = f(x) - f(o), \quad \forall f \in \mathcal{H}_E.$$

In this case, v_x satisfies $\Delta v_x = \delta_x - \delta_o$, and we make the choice $v_x(o) = 0$. The function $v_x : G^0 \to \mathbb{R}$ is called a *dipole*.

Example 3.1.

The dyadic tree.

- $\mathcal{A}$ = the alphabet of two letters, bits $\{0, 1\} \simeq \mathbb{Z}_2$.
- G^0: the set of all finite words in $\mathcal{A}$: $o = \emptyset$ = the empty word, $x = (a_1a_2 \cdots a_n) \in G^0$, $a_i \in \mathcal{A}$, a word of length n; $l(x) = n$.
- G^1 $\equiv$ the edges in the dyadic tree. If $x = \emptyset$, $Nbh(x) = \{0, 1\}$ two one-letter words. If $l(x) = n > 0$, $x = (a_1a_2 \cdots a_n)$, $Nbh(x) = \{(a_1 \cdots a_{n-1}), (x0), (x1)\}$. Set $x^* \equiv (a_1 \cdots a_{n-1})$.

- *Constant conductance.* This is the restriction $c \equiv 1$ on G^1. Then

$$(\Delta f)(o) = 2f(o) - f(0) - f(1), \quad \text{and}$$

$$(\Delta f)(x) = 3f(x) - f(x^*) - f(x0) - f(x1),$$

if $x \in G^0$, and $l(x) > 0$.

- *Paths in the tree.* If $x = (a_1 a_2 \cdots a_n) \in G^0$, there is a unique path $\gamma(x)$ from $\emptyset$ to x: the path is

$$\gamma(x) = \{(o, a_1), (a_1, (a_1 a_2)), \cdots ((a_1 \cdots a_{n-1}), x)\}$$

and consists of n edges.

- *Concatenation of words:* For $x = (a_1 a_2 \cdots a_n)$, $y = (b_1 b_2 \cdots b_m) \in G^0$. Set $z = z(xy) = (a_1 \cdots a_n b_1 \cdots b_m)$.

The dipoles (v_x) are indexed by $x \in G^0 \setminus (o)$, and $v_x(o) = 0$ where o is the chosen origin. If $G =$ the tree, then $o = \emptyset =$ the empty word.

Lemma 3.1.

[12] Let $x = (a_1 a_2 \cdots a_n)$, $a_i \in A$, $n = l(x)$; and $y = (b_1 b_2 \cdots b_m)$, $b_i \in A$, $m = l(y)$. Then

(i)

$$v_x(y) \equiv \begin{cases} 0 & \text{if } y = o, \\ 2^{n-m} \cdot (2^m - 1) - \frac{2^n - 1}{2} & \text{if } m \le n, \\ \frac{2^n - 1}{2} & \text{if } m > n. \end{cases}$$

(ii) $v_x \in \mathcal{H}_E$*, and* $\|v_x\|_E^2 = \frac{2}{3}(2^{2n} - 1)$.

(iii) $\langle v_x, v_y \rangle_E = \frac{2}{3}(2^{2\min(l(x)l(y))} - 1) = \#(\gamma(x) \cap \gamma(y))$*, for all* $x, y \in G^0 \setminus (o)$.

Proof. (i) By the uniqueness in Lemma 3.1, it is enough to prove that the function v_x in (i) satisfies $\langle v_x, f \rangle_E = f(x) - f(o)$ for all $f \in \mathcal{H}_E$, and therefore also

$$\Delta v_x = \delta_x - \delta_o; \tag{21}$$

and that (ii)-(iii) hold.

Specifically, we must prove that

$$\begin{aligned} &(\Delta v_x)(o) = -1, \\ &(\Delta v_x)(x) = 1, \text{ and} \\ &(\Delta v_x)(y) = 0, \text{ if } y \notin \{o, x\}. \end{aligned}$$

Each is a computation:

$$\begin{aligned} (\Delta v_x)(o) &= 2v_x(o) - v_x(0) - v_x(1) \\ &= 0 - (2 \cdot 2^{n-1} - (2^n - 1)) \\ &= 1 \\ &= \delta_o(o). \end{aligned}$$

And if $y \neq o$, but $m < n$, then

$$\begin{aligned} (\Delta v_x)(y) =& 3v_x(y) - v_x(y^*) - v_x(y0) - v_x(y1) \\ =& 3 \cdot 2^{n-m} \cdot (2^m - 1) - 2^{n-m+1} \cdot (2^{m-1} - 1) \\ & - 2 \cdot 2^{n-m-1} \cdot (2^{m+1} - 1) \\ =& 0. \end{aligned}$$

Finally, we compute the case $y = x$ as follows:

$$\begin{aligned} (\Delta v_x)(x) =& 3v_x(x) - v_x(x^*) - v_x(x0) - v_x(x1) \\ =& 3 \cdot (2^n - 1) - 2 \cdot (2^{n-1} - 1) - 2 \cdot (2^n - 1) \\ =& 0 - 3 + 2 + 2 = 1 \\ =& \delta_x(x) - \delta_o(x). \end{aligned}$$

We leave the case $m = l(y) > n$ to the reader.

(ii) A computation using (19) yields

$$\begin{aligned} \|v_x\|_E^2 &= \frac{1}{2} \sum_{m \le n} (2^{n-m})^2 \\ &= \frac{1}{2} \cdot 2^{2n} \cdot \left(\frac{1 - 2^{-2n}}{1 - 2^{-2}} \right) \\ &= \frac{2}{3}(2^{2n} - 1) \end{aligned}$$

proving (ii).

(iii) Suppose $m = l(y) < n = l(x)$, $x, y \in G^0 \setminus (o)$. From (20), we see that the contribution to $\langle v_x, v_y \rangle_E$ only includes words z with $l(z) \le m$.

The desired conclusion

$$\langle v_x, v_y \rangle_E = 2^{-2m} \#(\gamma(x) \cap \gamma(y))$$

follows as in (ii). The possibilities may be illustrated in Fig. 1 below.

□

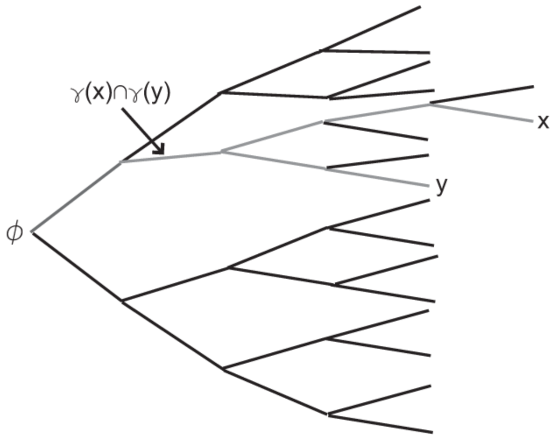

Figure 1. Dyadic tree-branching rules.

4. Specific transition operators

4.1. Transition on graphs

Let $G = (G^0, G^1)$ be a graph with conductance function $c : G^1 \to \mathbb{R}^+$, and transition probabilities

$$p(x, y) \equiv \frac{c(x, y)}{c(x)}, \quad \forall (x, y) \in G^1.$$

Note that $c(x)p(x, y) = p(x, y)c(y)$, which makes the corresponding p-random walk reversible.

Lemma 4.1.
Assume that $\#Nbh(x) < \infty$ for all $x \in G^0$. Set

$$(Tf)(x) \equiv \sum_{y \sim x} p(x, y)f(y),$$

and let (Z_n) be the random walk on G^0 with transition probabilities $p(x, y)$ on edges (xy) in G, i.e.,

$$\mathbb{P}(\{\gamma | Z_n(\gamma) = x, Z_{n+1}(\gamma) = y\}) = p(x, y) \quad \text{for } (xy) \in G^1$$

Let T be the transition operator, and for $\varphi \in l^1(G^0)$, set

$$\langle \varphi \rangle \equiv \sum_{x \in G^0} \varphi(x),$$

then for pairs of functions f_1 and f_2 on G^0, we have

$$\mathbb{E}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) = \langle T^n(f_1 \cdot Tf_2) \rangle$$

with f_1 and f_2 are restricted to make the last sum convergent.

Proof. Let f_1, f_2 be a pair of functions (real valued) on G^0 such that the pointwise product $f_1 \cdot (Tf_2)$ is in $l^1(G^0)$. Then for $n \in \mathbb{N}_0$, we now compute the Z_n-expectations: For the $\mathbb{P}$-integration on path space Ω, we have:

$$\begin{aligned}
\mathbb{E}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) &= \int_\Omega (f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1}) d\mathbb{P} \\
&= \sum_{\substack{x_0 \\ \text{such that } x_{i-1} \sim x_i}} \sum_{x_1} \cdots \sum_{x_{n+1}} p(x_0, x_1)p(x_1, x_2) \cdots p(x_n, x_{n+1}) f_1(x_n) f_2(x_{n+1}) \\
&= \sum_{x_0} \sum_{x_1} \cdots \sum_{x_n} p(x_0, x_1)p(x_1, x_2) \cdots p(x_{n-1}, x_n) f_1(x_n)(Tf_2)(x_n) \\
&= \sum_{x_0 \in G^0} T^n(f_1 \cdot Tf_2)(x_0) \\
&= \langle T^n(f_1 \cdot Tf_2) \rangle.
\end{aligned}$$

□

Theorem 4.1.
Let (G, c) be a graph with conductance $c : G^1 \to \mathbb{R}_+$. Assume that $\sharp Nbh(x) < \infty$ for all $x \in G^0$, when $Nbh(x) \equiv \{y \in G^0 | y \sim x\}$. Set

$$p(x, y) \equiv \frac{c(x, y)}{c(x)} \quad \text{and} \quad (Tf)(x) \equiv \sum_{y \sim x} p(x, y)f(y).$$

Set

$$l^1(G^0,\mu_c)=\{f:G^0\to\mathbb{R}|x\to c(x)f(x)\in l^1(G^0)\},\quad \textit{and}\quad \langle f\rangle_c\equiv\sum_{x\in G^0}c(x)f(x).$$

Let $\mathbb{P}^{(c)}=\mathbb{P}^{(\mu_c)}$ *be the cylinder path-measure on*

$$\Omega\equiv\{(x_0x_1x_2\cdots)|x_i\in G^0,x_{i-1}\sim x_i,i\in\mathbb{N}\},$$

where we use μ_c *in the first variable* x_0*, and counting measure on the remaining variables. Then*

$$\mathbb{E}^{(\mu_c)}((f_1\circ Z_n)\cdot(f_2\circ Z_{n+1}))=\langle f_1\cdot Tf_2\rangle_c.$$

Proof.

$$\begin{aligned}\mathbb{E}^{(\mu_c)}((f_1\circ Z_n)\cdot(f_2\circ Z_{n+1}))&=\underset{\text{such that } x_{i-1}\sim x_i}{\sum_{x_0}\sum_{x_1}\cdots\sum_{x_{n+1}}}c(x_0)p(x_0,x_1)p(x_1,x_2)\cdots p(x_n,x_{n+1})f_1(x_n)f_2(x_{n+1})\\&=\sum_{x_0}\sum_{x_1}\cdots\sum_{x_n}c(x_0)p(x_0,x_1)p(x_1,x_2)\cdots p(x_{n-1},x_n)f_1(x_n)(Tf_2)(x_n)\\&=\sum_{x_0}c(x_0)T^n(f_1\cdot Tf_2)(x_0)\\&=\langle T^n(f_1\cdot Tf_2)\rangle_c\\&=\langle f_1\cdot Tf_2\rangle_c.\end{aligned}$$

In the multiple summations $\sum_{x_0}\sum_{x_1}\cdot\sum_{x_{n+1}}$, it is just the first $\sum_{x_0}$-summation that is possibly infinite; in case the vertex-set G^0 is infinite. Note that the combined summations in the beginning of the proof contribute the integration over the set Ω of all infinite paths $\gamma=(x_0x_1x_2\cdots)$ specified by $x_0\sim x_1$, $x_1\sim x_2$, $x_2\sim x_3$, $\cdots$, at each step, moving from x_i to the next variable, note that x_{i+1} ranges over the finite set $Nbh(x_i)$. For more details on this point, see (22), below. In the last step, we used the following formula which is valid on $l^1(\mu_c)$:

$$\langle T\varphi\rangle_c=\langle\varphi\rangle_c,\quad \varphi\in l^1(\mu_c).\tag{22}$$

We prove (22):

$$\langle T\varphi\rangle_c=\sum_{x\in G^0}c(x)\sum_{y\sim x}p(x,y)\varphi(y)=\sum_{y\in G^0}\varphi(y)\sum_{x\sim y}c(x,y)=\sum_{y\in G^0}\varphi(y)c(y)=\langle\varphi\rangle_c.$$

□

4.2. Transfer operators

In Sec. 2, we showed that a stochastic process $(Z_n)_{n\in\mathbb{N}_0}$ on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ induces a transfer operator T. The derivation of T is then essentially canonical. Here, the strategy will be reversed; but now, starting with T, there is a variety of choices of associated processes $(Z_n)_{n\in\mathbb{N}_0}$.

4.2.1. Setting

Let S be a compact Hausdorff space. Let $(S,\mathcal{B})_S,\mu)$ be a Borel probability measure space, and let $p:S\times S\to\mathbb{R}_{\geq 0}$ be a continuous function such that

$$\int_S p(x,y)d\mu(y)\equiv 1\quad \mu\text{ a.e. } x.\tag{23}$$

Set

$$(Tf)(x)\equiv\int_S p(x,y)d\mu(y)\quad\text{for all } f\in L^\infty(S).\tag{24}$$

Set

$$\Omega \equiv \Omega_p = \{\gamma = (x_0 x_1 x_2 \cdots) | x_i \in S, \\ \text{s.t. } p(x_{i-1}, x_i) > 0\}, \quad (25)$$

so an infinite path-space with path transitions governed by te function p.
Let $\mathbb{P} = (\mathbb{P}_p)$ be the associated cylinder measure on Ω_p as defined in Sec. 2. For $n \in \mathbb{N}_0$ and $\gamma = (x_0 x_1 x_2 \cdots) \in \Omega_p$, set

$$Z_n(\gamma) \equiv x_n; \quad i.e., Z_n : \Omega_p \to S \quad (26)$$

is an S valued random variable for all $n \in \mathbb{N}_0$.

Theorem 4.2.
Let $p : S \times S \to \mathbb{R}_{\geq 0}$ be as stated in (23) above. Let T be the transfer operator (24). Then the stochastic process $(Z_n)_{n \in \mathbb{N}_0}$ in (26) satisfies

$$\mathbb{E}^{(p)}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) = \int_S (T^n(f_1 \cdot Tf_2))(x) d\mu(x) \\ \text{for all } f_1, f_2 \in L^\infty(S). \quad (27)$$

Proof. The details in the computation for (27) follow those in Sec. 2, but the reasoning is now reversed. Indeed,

$$\begin{aligned}\mathbb{E}^{(p)}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) &= \int_S \int_S \cdots \int_S p(x_0, x_1) \cdots p(x_n, x_{n+1}) f_1(x_n) f_2(x_{n+1}) d\mu(x_0) d\mu(x_1) \cdots d\mu(x_{n+1}) \\ &= \int_S \int_S \cdots \int_S p(x_0, x_1) \cdots p(x_{n-1}, x_n) f_1(x_n)(Tf_2)(x_n) d\mu_0(x_0) d\mu(x_1) \cdots d\mu(x_n) \\ &= \int_S (T^n(f_1 \cdot Tf_2))(x) d\mu(x).\end{aligned}$$

□

Definition 4.1.
Let T be a transition operator satisfying the conditions (23) and (24), and suppose there is a Perron-Frobenius measure μ_0 on S, *i.e.*,

$$\mu_0 \circ T = \mu_0. \quad (28)$$

We say that T is *ergodic* if there is only one probability measure μ_0 on $(S, \mathcal{B}_S)$ which solves (28).
If T is ergodic, and μ_0 is the (unique) Perron-Frobenius measure, then it follows from the Pointwise Ergodic Theorem that for all $f \in L^\infty(S)$, the limit

$$\lim_{n \to \infty} T^n(f) = \mu_0(f)\mathbb{1}, \quad (29)$$

pointwise a.e. exits on S, where $\mathbb{1}$ denotes the constant function 1 on S.

Corollary 4.1.
Let p, T, S, $\mathcal{B}_S$, μ, and (Z_n) satisfy the conditions of the theorem. Further assume T is ergodic with Perron-Frobenius measure μ_0. Then

$$\lim_{n \to \infty} \mathbb{E}^{(p)}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) = \mu_0(f_1 \cdot Tf_2) \quad (30)$$

is satisfied for all $f_1, f_2 \in L^\infty(S)$.

Proof. To verify (30), note that $\mathbb{E}^{(p)}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1}))$ is already computed in (27) in the theorem.
Since μ is a probability measure, the conclusion (30) now follows from (29), *i.e.*, form an application of the Ergodic Theorem. □

4.3. Transition on solenoids

Let (S, μ) be a measure space, $\sigma : S \to S$ an endomorphism as specified in Sec. 2. Let $\Omega \equiv X_\sigma(S)$ be the corresponding solenoid. Let $W : S \to [0, 1]$ be a function satisfying

$$\sum_{y, \sigma(y) = x} W(y) = 1; \quad (31)$$

and let $\mathbb{P} = \mathbb{P}_{\mu, \sigma, W}$ be the corresponding path measure.

Lemma 4.2.
For the solenoid set $Z_n : \Omega \to S$, $Z_n(x_0, x_1, x_2, \cdots) = x_n$, and $(Tf)(x) = \sum_{y, \sigma(y) = x} W(y) f(y)$, for $x \in S$. Suppose T has a Perron-Frobenius measure μ_0. Then $(Z_n)_{n \in \mathbb{N}_0}$ is stationary with transition operator T.

Proof. Let f_1, f_2 be a pair of functions on S satisfying the conditions listed above. For the $\mathbb{P}$-integration on path space $\Omega(= X_\sigma(S))$ we then have:

$$\begin{aligned}
\mathbb{E}^{(\mu_0)}((f_1 \circ Z_n) \cdot (f_2 \circ Z_{n+1})) &= \int_S \sum_{\substack{x_1 \\ \sigma(x_1)=x_0}} \sum_{\substack{x_2 \\ \sigma(x_2)=x_1}} \cdots \sum_{\substack{x_{n+1} \\ \sigma(x_{n+1})=x_n}} W(x_1)W(x_2)\cdots W(x_{n+1}) f_1(x_n) f_2(x_{n+1}) d\mu_0(x_0) \\
&= \int_S \sum_{\substack{x_1 \\ \sigma(x_1)=x_0}} \sum_{\substack{x_2 \\ \sigma(x_2)=x_1}} \cdots \sum_{\substack{x_n \\ \sigma(x_n)=x_{n-1}}} W(x_1)W(x_2)\cdots W(x_n) f_1(x_n)(Tf_2)(x_n) d\mu_0(x_0) \\
&= \int_S (T^n(f_1 \cdot Tf_2))(x_0) d\mu_0(x_0) \\
&= \mu_0(T^n(f_1 \cdot Tf_2)) = \mu_0(f_1 \cdot Tf_2) \\
&= \langle f_1, Tf_2 \rangle_{L^2(\mu_0)}.
\end{aligned}$$

□

4.4. Encodings

Let $G = (G^0, G^1)$ be a graph where we write G^0 for the vertices and G^1 for the edges. Let S be a set. We say that G yields an encoding of the points in S if there are mappings

$$\tau^0 : G^0 \to S, \quad \text{onto, and} \tag{32}$$

$$\tau^1 : G^0 \to \text{Functions } (S \to S), \tag{33}$$

such that for every $e = (x, y) \in G^1$ we have

$$\tau^0(y) = \tau^1(e)\tau^0(x). \tag{34}$$

Examples

$G =$ the binary tree,

$$\begin{aligned} S &= \mathbb{N}_0 = \{0, 1, 2, \cdots\} \\ &= \{\sum_{k=0}^{\text{Finite}} x_k 2^k | x_k \in \{0, 1\}\}. \end{aligned} \tag{35}$$

If $n \in \mathbb{N}_0$ is given the finite word $(x_0x_1x_2\cdots)$ in (4.6) is computed from the Euclidean algorithm for division with 2.
Points in G^0 are represented by the empty word o, and by all finite words $w = (x_0x_1\cdots x_p)$. Set

$$\tau^0(w) = \sum_{k=0}^{p} x_k 2^k = n \in \mathbb{N}_0. \tag{36}$$

Starting with $w = (x_0x_1\cdots x_p) \in G^0$, the three neighbors are $(w0), (w1)$, and $w^* \equiv (x_0x_1\cdots x_{p-1})$ truncation, see Fig. 3.

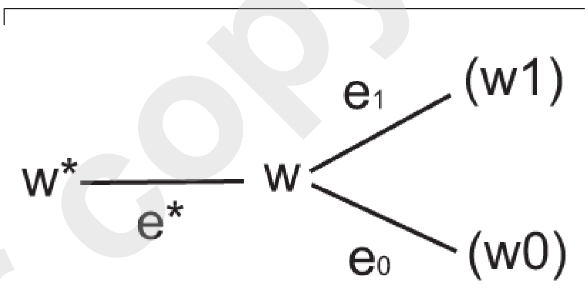

Figure 2. Three nearest neighbors and then associated these edges e_0, e_1 and e^*.

Set

$$\begin{cases} \tau^1(e_0) \equiv n \mapsto n; & \text{see (36);} \\ \tau^1(e_1) \equiv n \mapsto n + 2^{p+1}; & \text{and} \\ \tau^1(e^*) \equiv n \mapsto \sum_{k=0}^{p-1} x_k 2^k. \end{cases} \tag{37}$$

Note that in this example, there is an additional pair of mappings $\mathbb{N}_0 \to \mathbb{N}_0$

$$\begin{cases} \sigma^0(n) = 2n, \\ \sigma^1(n) = 2n + 1, \end{cases} \tag{38}$$

corresponding to the encoding mappings:

$$\begin{cases} \sigma_0 : (x_0x_1\cdots x_p) \mapsto \underbrace{(0x_0x_1\cdots x_p)}_{\text{one step longer}}, \\ \sigma_1 : (x_0x_1\cdots x_p) \mapsto (1x_0x_1\cdots x_p). \end{cases} \tag{39}$$

Remark 4.1.
The same construction works mutatis mutandis with N'adic scaling rather than the dyadic representation of

points in $\mathbb{N}_0$. Moreover, in the representation

$$n = \sum_{k=0}^{p} x_k N^k, \tag{40}$$

the choices for x_k may be from any complete set of residues modulo N, *i.e.*, points in $\mathbb{N}_0/N\cdot\mathbb{N}_0$, or $\mathbb{Z}/N\mathbb{Z}$ = the cyclic group of order N. The residues $\{0, 1, \cdots, N-1\}$ is only one choice of many.

Encoding of $\mathbb{Z}$

The representation used in (36) above works for $\mathbb{Z}$ as well, but with the following modification:

$$\tau^0(x_0x_1x_2\cdots x_p) \equiv -2^p + \sum_{k=0}^{p} x_k 2^k. \tag{41}$$

Explanation:

$$\begin{aligned}\tau^0(\underbrace{111\cdots 1}_{p+1 \text{ times}}) &= -2^p + \sum_{k=0}^{p} x_k 2^k, \quad \text{with } x_k = 1, 0 \le k \le p\\ &= -2^p + 2^{p+1} - 1\\ &= 2^p - 1.\end{aligned}$$

Hence, with this convention we arrive at an encoding of $\mathbb{Z}$.

Graphs vs compactification:

In the examples, we represent points in the vertex sets G^0 on a graph G by finite words in a specific finite alphabets. A choice of compactification Ω of G^0 is the set of infinite paths γ, *i.e.*, $\gamma = (x_0x_1x_2\cdots)$ where $x_i \in G^0$, and $(x_{i-1}, x_i) \in G^1$ for all $i \in \mathbb{N}$.

In each of the examples we present, we build measure $\mathbb{P}$ on the compactifications Ω with use of Kolmogorov's extension principle. This is a projective limit construction which proceeds in three steps [11]:

(i) First specify $\mathbb{P}$ only on finite words, *i.e.*, on cylinder sets over G^0

(ii) Check that the prescription of $\mathbb{P}$ on cylinders is consistent.

(iii) With Kolmogorov's theorem than extend $\mathbb{P}$ to the Borel σ-algebra of subsets in Ω generated by the cylinder-sets [11, 15].

Definition 4.2.

In later applications, the following two cases for $\mathbb{P}$ will play a role: Consider the subset Ω_{Fin} in Ω consisting of paths $\gamma = (x_0x_1x_2\cdots)$ which terminate in infinite repetitions, *i.e.*, $\gamma \in \Omega_{\text{Fin}} \Leftrightarrow \exists\, n$ such that $x_i = x_n\ \forall\, i > n$. The measure $\mathbb{P}$ is said to be *tight* if and only if $\mathbb{P}(\Omega_{\text{Fin}}) = 1$. Alternatively, $\mathbb{P}(\Omega_{\text{Fin}}) < 1$.

Examples resumed:

Wavelets. We adopt the standard terminology for dyadic wavelets in $L^2(\mathbb{R})$, specifically φ for a choice of scaling function; see [11]. Let $(a_k)_{k\in\mathbb{Z}}$ represent a wavelet filter, *i.e.*, satisfying the following three conditions:

$$\sum_{k\in\mathbb{Z}} \overline{a}_k a_{k+2l} = \frac{1}{2}\delta_{0,l}, \tag{42}$$

$$\sum_{k\in\mathbb{Z}} a_k = 1, \quad \text{and} \tag{43}$$

$$\varphi(x) = 2\sum_{k\in\mathbb{Z}} a_k \varphi(2x-k). \tag{44}$$

The function φ is in $L^2(\mathbb{R})$ and

$$\int_{\mathbb{R}} \varphi(x)dx = 1 \tag{45}$$

is a chosen normalization.

Let $\widehat{\varphi}$ be the $\mathbb{R}-$ Fourier transform.

The following result is from [11]. Let $\Omega \equiv$ the set of all infinite words, and view Ω as a compactification of the vertex set G^0 of all finite dyadic words.

Lemma 4.3.

For every $t \in \mathbb{R}$, there is a measure $\mathbb{P}_t$ on Ω such that

$$\mathbb{P}_t(x_0x_1\cdots x_p) = \left|\hat{\varphi}(t + \tau^0(x_0x_1\cdots x_p))\right|^2, \tag{46}$$

where $\tau^0 : G^0 \to \mathbb{Z}$ is the encoding of (41).

Lemma 4.4.

(See [11].)

(a) Consider the process (Z_n) in $(\Omega, \mathbb{P}_t)$ from (46) with

$$Z_n\underbrace{(x_0x_1x_2\cdots)}_{\textit{infinite word}} \equiv x_n \in \{0, 1\}.$$

Then there is a transfer operator T such that the process is T-stationary.

(b) *Let*

$$W(e^{it}) \equiv \widetilde{W}(t) = \left|\sum_{k\in\mathbb{Z}} a_k e^{ikt}\right|^2, \tag{47}$$

where functions W on $\mathbb{T}$ are identified with 2π-periodic functions $\widetilde{W}$ on $\mathbb{R}$, and where (a_k) is some wavelet filter as in (42)-(44). The transfer operator T is then given by

$$(T_W f)(t) = W(\frac{t}{2})f(\frac{t}{2}) + W(\frac{t}{2}+\pi)f(\frac{t}{2}+\pi).$$

We say that W has scaling-degree 2.

Following (38), let a transition from n to $n+1$ be given by a choice of $x \in \{0,1\}$.

Then

$$\mathbb{E}_t(Z_n Z_{n+1}) = \widetilde{W}(t + x\pi). \tag{48}$$

Proposition 4.1.

Let $\varphi \in L^2(\mathbb{R})$ satisfying (44), and suppose $\|\varphi\|_2 \le 1$. Let Ω be the compactification derived from the encoding τ^0 of $\mathbb{Z}$ in (41) and let $t \in (-\pi,\pi]$. Let $\mathbb{P}_t$ be the measure on Ω from (46).

Part I

Then the following affirmations are equivalent:

(a) *The translates $\{\varphi(\cdot - k) | k \in \mathbb{Z}\}$ form an orthonormal family in $L^2(\mathbb{R})$.*

(b) *The measures $\mathbb{P}_t$ are tight measures on Ω for all t.*

(c) *$\sum_{n\in\mathbb{Z}} |\widehat{\varphi}(t+n)|^2 = 1$ for all $t \in \mathbb{R}$.*

Part II

If the measures $\mathbb{P}_t$ are not tight, then the translates $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ still form a Parseval frame for the closed subspace $V(\varphi)$ they span, i.e., we have the identity

$$\sum_{k\in\mathbb{Z}} \left|\int_{\mathbb{R}} \overline{\varphi}(x-k)f(x)dx\right|^2 = \int_{\mathbb{R}} |f(x)|^2 dx \quad \text{for all } f \in V(\varphi).$$

Proof. See [11]. □

Definition 4.3.

Functions W on $(-\pi,\pi]$ arising as in (47) for a system of wavelet coefficients $(a_k)_{k\in\mathbb{Z}}$ (44), are called *wavelet filters*. A wavlet filter W is said to be *low-pass* if $\mu_0 \equiv \delta_0$, *i.e.*, the Dirac measure at $\theta = 0$, is a Perron-Frobenius measure for T_W.

In general, if W is a Lipschitz function, it is known that T_W has a Perron-Frobenius measure [3].

Example 4.1.

[5] Set

$$W_F(z) \equiv \frac{1}{6}|1+z^2| \quad \text{for } z = e^{i\theta}. \tag{49}$$

Then W_F is a wavelet-filter under scaling by 3, but it is *not* a low-pass filter.

Indeed, the following scaling law holds for W_F:

$$\sum_{w^3=z} W_F(w) = 1, \quad \forall z = e^{i\theta} \in \mathbb{T}^1.$$

We say that W_F has scaling degree 3.

It is proved in [5] that W_F induces a wavelet representation on an L^2-space built from the middle-third-Canter construction, "Cantor-dust" CD_3 in $\mathbb{R}$ with Hausdorff measure $\mathcal{H}^\alpha$, $\alpha = \frac{\ln 2}{\ln 3}$, *i.e.*, on L^2(Cantor dust, $\mathcal{H}^\alpha$).

Cantor dust CD_3

The points $x \in CD_3 \subset \mathbb{R}$ are encoded by

$$x = a_{-k}3^k + a_{-k+1}3^{k-1} + \cdots + a_0 + \sum_{i=0}^{\infty} \frac{a_i}{3^i},$$

where k varies in $\mathbb{N}_0$, and where $a_j \in \{0,1,2\}$ for $j \in \mathbb{Z}$ such that $-k \le j$; but where a_j attains the value 1 only for at most a finite number of places.

The Perron-Frobenius measure μ_0 for T_{W_F} is singular with support $(\mu_0) = \mathbb{T}$.

5. Reprocity rule for the spectrum

In the previous section we saw that a wide class of processes are governed by a transfer operator T. If the process in question takes places on a graph $G = (G^0, G^1)$ with conductance c, then harmonic analysis on G is phrased in terms of a Laplace operator Δ_c as follows:

$$(\Delta_c f)(x) = \sum_{y\sim x} c(x,y)(f(x) - f(y)), \quad \text{for } x \in G^0.$$

Lemma 5.1.

Let (G,c) and Δ_c be as above. Set $p(x,y) = \frac{c(x,y)}{c(x)}$ for $(x,y) \in G^1$ and let

$$(T_c f)(x) = \sum_{y\sim x} p(x,y)f(y),$$

then

$$(\Delta_c f)(x) = c(x)\{f(x) - (T_c f)(x)\}.$$

And conversely,

$$(T_c f)(x) = f(x) - \frac{1}{c(x)}(\Delta_c f)(x).$$

Proof. Left to the reader. □

Because of reference to harmonic analysis, we present the results in this section in terms of Δ_c, but the lemma makes a translation between Δ_c and T_c immediate: For example, a function f on G^0 satisfies $\Delta_c f = 0$ if and only if $T_c f = f$. Solution f to either one of these equations are called *harmonic*.

Definition 5.1.

Let $\mathcal{H}$ be a Hilbert space, and $\mathcal{D}$ a dense linear subspace. An operator Δ defined on $\mathcal{D}$ is said to be *formally selfadjoint* if and only if

$$\langle \Delta u, v\rangle = \langle u, \Delta v\rangle$$

holds for all $u, v \in \mathcal{D}$.

A further advantage of Δ_c over T_c is that Δ_c is formally selftadjoint, (while T_c is not!).
When we say that Δ_c is formally selfadjoint, this applies to either one of the two Hilbert spaces $l^2(G^0)$, and $\mathcal{H}_E \equiv$ the energy Hilbert space.
In the case of $\mathcal{H}_E$, we take for $\mathcal{D}$ the linear span of the family $\{v_x | x \in G^0\} \subset \mathcal{H}_E$; see Lemma 5.2 and 5.3.
We continue the setup from the previous section: $G = (G^0, G^1)$ a fixed graph with vertices G^0 and edges G^1. Let $c : G^1 \to \mathbb{R}_+$ be a fixed conductance function. Let $\Delta = \Delta_c$ be the Laplace operator. Fix an origin o in G^0, and let $\{v_x\}_{x\in G^0\setminus(0)}$ be the system of dipoles.

Lemma 5.2.

[12] (Reproducing Kernel) The system $\{v_x\}_{x\in G^0\setminus(o)}$ forms a reproducing kernel in the sense:

$$\langle v_x, f\rangle_E = f(x) - f(o) \quad \text{for all } f \in \mathcal{H}_E, \tag{50}$$

where $\mathcal{H}_E$ is the energy Hilbert space.

Proof. The existence of $\{v_x\}$ is established with an application of Riesz's lemma: If $x \in G^0$, there is a path $\gamma(x) = x_0 \to x_1 \to \cdots \to x_n$, $e_i = (x_i\ x_{i+1}) \in G^1$, (generally not unique) such that $x_0 = 0$ and $x_n = x$.
By Cauchy-Schwarz, we get

$$|f(x) - f(o)|^2 \le \sum_i \frac{1}{c(e_i)} \|f\|_E^2. \tag{51}$$

Riesz's lemma applied to $\mathcal{H}_E$, then yields $\exists v_x \in \mathcal{H}_E$ such that (50) is satisfied.

We claim that v_x satisfies the dipole equation

$$\Delta v_x = \delta_x - \delta_o, \quad x \in G^0 \setminus (o). \tag{52}$$

This implies (52), and if $\Delta h = 0$, then $w_x \equiv v_x + h$ solves (52) as well; and vice versa. □

Lemma 5.3.

[12] Let $\mathcal{D}_0 \equiv span_{\mathbb{C}}\{\delta_x\}_{x\in G^0}$, and $\mathcal{D}_E \equiv span_{\mathbb{C}}\{v_x\}_{x\in G^0\setminus(o)}$.

By "span" we mean finite complex linear combinations, so we consider all finite summations

$$\mathcal{D}_0 = \left\{\sum_x a_x \delta_x\right\}, \quad \text{and} \quad \mathcal{D}_E = \left\{\sum_x b_x v_x\right\}, \tag{53}$$

where $\{a_x\}$ and $\{b_x\}$ denote finite systems of scalars, a_x, $b_x \in \mathbb{C}$.
Then Δ yields a density defined hermitian (*i.e.*, formally selfadjoint) operator in each of the Hilbert spaces $l^2(G^0)$ and $\mathcal{H}_E$.
Specifically, $\mathcal{D}_0$ is dense in $l^2(G^0)$ and

$$\langle u, \Delta v\rangle_{l^2} = \langle \Delta u, v\rangle_{l^2}, \quad \forall u, v \in \mathcal{D}_0. \tag{54}$$

Moreover, $\mathcal{V}$ is dense in $\mathcal{H}_E$, and

$$\langle u, \Delta v\rangle_E = \langle \Delta u, v\rangle_E, \quad \forall u, v \in \mathcal{D}_E. \tag{55}$$

Proof. The symmetry property (54) is immediate from the definition of Δ.
We now prove (55): Since both sides in (54) are sesquilinear, it is enough, by (53), to prove

$$\langle v_x, \Delta v_y\rangle_E = \langle \Delta v_x, v_y\rangle_E, \quad \forall x, y \in G^0 \setminus (o). \tag{56}$$

We have

$$\begin{aligned}
\langle v_x, \Delta v_y\rangle_E &\underset{\text{by } (52)}{=} \langle v_x, \delta_y - \delta_0\rangle_E \\
&\underset{\text{by } (50)}{=} (\delta_y - \delta_0)(x) - (\delta_y - \delta_0)(o) \\
&= \delta_x(y) + 1 \\
&\underset{\text{by symmetry}}{=} \langle \delta_x - \delta_0, v_y\rangle_E \\
&\underset{\text{by } (52)}{=} \langle \Delta v_x, v_y\rangle_E,
\end{aligned}$$

which is the desired Eq. (56). □

5.1. Two Hilbert spaces

Let $G = (G^0, G)$ be as above; and let $c : G^1 \to \mathbb{R}_+$ be a fixed conductance function. Let Δ and T be the corresponding operators, $\Delta = \Delta_c$ the Laplace operator, and

$$(Tf)(x) = f(x) - \frac{1}{c(x)}(\Delta f)(x), \quad x \in G^0. \tag{57}$$

Pick a fixed $o \in G^0$, and let $(v_x)_{x\in G^0\setminus(o)}$ be the corresponding reproducing kernet.

It is important to understand the two operators in the two Hilbert spaces $l^2(G^0)$ and $\mathcal{H}_E$. By (57), it is enough to consider just Δ.

As an operator in $l^2(G^0)$, the operator Δ has as its domain

$$\begin{aligned}\mathcal{D}_0 \equiv&\ \text{all finite linear combinations of } \{\delta_x\}_{x\in G^0}\\ =&\ \text{span}\ \{\delta_x\}_{x\in G^0};\end{aligned}$$

while the domain in $\mathcal{H}_E$ is

$$\mathcal{D}_E \equiv \text{span}\ \{v_x | x \in G^0 \setminus (o)\}.$$

Theorem 5.1. *(a) The domains in l^2 and in $\mathcal{H}_E$:*

(i) $\mathcal{D}_0$ is a dense subspace in $l^2(G^0)$; and

(ii) $\mathcal{D}_E$ is a dense subspace in $\mathcal{H}_E$.

(iii) If $\sharp Nbh(x) < \infty$ for all $x \in G^0$, then Δ maps $\mathcal{D}_0$ into itself; and Δ_E maps $\mathcal{D}_E$ into itself.

(b) For all vectors $\varphi, \psi \in \mathcal{D}_0$, we have:

(i)

$$\langle \varphi, \Delta\varphi\rangle_{l^2} = \sum_{x\in G^0} c(x)|\varphi(x)|^2 - \sum_{x}\sum_{\substack{y\\ x\sim y}} c(x,y)\overline{\varphi}(x)\varphi(y);$$

(ii) $\langle \varphi, \Delta\varphi\rangle_{l^2} \geq 0$; and

(iii) $\langle \varphi, \Delta\psi\rangle_{l^2} = \langle \Delta\varphi, \psi\rangle_{l^2}$.

(c) For all vectors $\varphi, \psi \in \mathcal{D}_E$, we have:

(i)

$$\langle \varphi, \Delta\varphi\rangle_{\mathcal{H}_E} = \sum_{x\in G^0\setminus(o)} |(\Delta\varphi)(x)|^2 + \left|\sum_{x\in G^0\setminus(o)} (\Delta\varphi)(x)\right|^2;$$

(ii) $\langle \varphi, \Delta\varphi\rangle_{\mathcal{H}_E} \geq 0$; and

(iii) $\langle \varphi, \Delta\psi\rangle_{\mathcal{H}_E} = \langle \Delta\varphi, \psi\rangle_{\mathcal{H}_E}$.

Proof. The proof of (b)(ii) is a sequence of steps with repeated application of Cauchy-Schwarz's inequality. The proof of (a)(i) is an application of the last equation in the proof of Lemma 5.3. □

Remark 5.1.
The operator Δ_{l^2} in l^2, or Δ_E in $\mathcal{H}_E$, may be bounded or unbounded. In all cases Δ_{l^2} is essentially selfadjoint in l^2 [12]; but Δ_E may have defect-subspaces.

5.2. Dichotomy

Remark 5.2.
[12] For the graph system $(G, c) = (tree, \mathbb{1})$ the Laplace operator $(\Delta, \mathcal{D}_0)$ is bounded and selfadjoint in $l^2(G^0)$. For the energy Hilbert space $\mathcal{H}_E(tree)$, $(\Delta, \mathcal{D}_E)$ is an *unbounded* Hermitian operator. In fact, Δ is *not* essentially selfadjoint on Δ; *i.e.*, $(\Delta, \mathcal{D}_E)$ has a infinite family of distinct selfadjoint extensions in the Hilbert space $\mathcal{H}_E$.

Lemma 5.4.
Let $\mathcal{H}\langle\cdot,\cdot\rangle$ be a complex Hilbert space, and let $\mathcal{D}$ be a dense linear subspace in $\mathcal{H}$.
Let L be a closed Hermitian operator defined on $\mathcal{D}$, i.e., L is linear and satisfies

$$\langle u, Lv\rangle = \langle Lu, v\rangle \quad \forall u, v \in \mathcal{D}. \tag{58}$$

Then the spectrum of Δ is the closure of the set

$$NS(L) \equiv \left\{ \frac{\langle u, Lu\rangle}{\|u\|^2} \,\middle|\, u \in \mathcal{D}\setminus(o)\right\}. \tag{59}$$

Proof. The Hermitian property (58) implies that the spectrum of L is contained in $\mathbb{R}$.

Now suppose $\lambda_0 \in \mathbb{R}$, and that

$$\text{dist}(\lambda_0, NS(L)) = \epsilon_1 > 0. \tag{60}$$

We will show that λ must then be in

$$\begin{aligned}\mathbb{R}\setminus spec(L) =&\ \text{the complement of the spectrum}\\ =&\ \text{the resolvent set.}\end{aligned}$$

Let $u \in \mathcal{D}\setminus(o)$. Then

$$\|\lambda_0 u - Lu\|^2 = \lambda_0^2\|u\|^2 - 2\lambda_0\langle u, Lu\rangle + \|Lu\|^2.$$

Setting $x_1 \equiv \frac{\langle u, Lu \rangle}{\|u\|^2} \in NS(L)$, we get

$$\begin{aligned} \|\lambda_0 u - Lu\|^2 &= \|u\|^2 \cdot (\lambda_0 - x_1)^2 - \|u\|^2 x_1^2 + \|Lu\|^2 \\ &\underset{by\ (60)}{\geq} \|u\|^2 \cdot \epsilon_1^2 + \|Lu\|^2 - \frac{\langle u, Lu \rangle^2}{\|u\|^2} \\ &\geq \|u\|^2 \cdot \epsilon_1^2, \end{aligned} \tag{61}$$

where we used Schwarz' inequality in the last step; viz.,

$$\langle u, Lu \rangle^2 \leq \|u\|^2 \cdot \|Lu\|^2;$$

or

$$\|Lu\|^2 - \frac{\langle u, Lu \rangle^2}{\|u\|^2} \geq 0.$$

By virtue of the inequality (2.11), we may define an operator

$$R_0 = R(\lambda_0) : range(\lambda_0 I - L) \longrightarrow \mathcal{H}$$

by

$$R_0(\lambda u - Lu) = u. \tag{62}$$

Extend R_0 by setting it $= 0$ on the ortho-complement

$$(range(\lambda_0 I - L))^{\perp} = N(\lambda_0 - L^*). \tag{63}$$

Here L^* denotes the adjoint operator.
From (62), we calculate that $R_0 : \mathcal{H} \to \mathcal{H}$ defines a bounded inverse to $\lambda_0 I - L$, and so $\lambda_0 \in$ resolvent(L); and conversely. □

Let $\{v_x\}_{x \in G^0 \setminus (0)}$ be the system of dipoles, and set

$$M \equiv (\langle v_x, v_y \rangle_E) \tag{64}$$

viewed as a Hermitian matrix, x = row index, y = column index.
If $\xi = (\xi_x) \in \mathcal{F} \subset l^2(G^0)$, set

$$(M\xi)_x = \sum_y M_{x,y} \xi_y, \tag{65}$$

matrix multiplication, where

$$M_{x,y} \equiv \langle v_x, v_y \rangle_E.$$

Then M is a density defined Hermitian operator in $l^2(G^0)$.

Theorem 5.2.
Let (G, μ) be given and let Δ be the corresponding density defined Hermitian operator in $\mathcal{H}_E$. Then

$$spec_{\mathcal{H}_E}(\Delta) \subset [0, \infty) \tag{66}$$

and

$$spec_{\mathcal{H}_E}(\Delta) = (spec_{l^2}(M))^{-1}, \tag{67}$$

where we use the charactors $\frac{1}{0} = \infty$, and $\frac{1}{\infty} = 0$. Moreover,

$$(spec_{l^2}(M))^{-1} = \{1/\lambda | \lambda \in spec_{l^2}(M)\}. \tag{68}$$

Proof. For $(\xi_x) \in \mathcal{F}$, set

$$u \equiv \sum_{x \in G^0 \setminus (0)} \xi_x v_x. \tag{69}$$

Then $u \in \mathcal{V}$, and

$$\begin{aligned} \langle u, \Delta u \rangle_{\mathcal{H}_E} &= \sum_x \sum_y \overline{\xi}_x \xi_y \langle v_x, \Delta v_y \rangle_E & (70) \\ &= \sum_x \sum_y \overline{\xi}_x \xi_y (\delta_x(y) + 1) & (71) \\ &= \sum_x |\xi_x|^2 + \left| \sum_x \xi_x \right|^2 \geq 0. & (72) \end{aligned}$$

Since vectors in $\mathcal{H}_E$ are equivalence classes modulo the constant function on G^0, we may add the restriction $\sum_x \xi_x = 0$ in (69), and the operator Δ will be unchanged. The modified equation (2.22) then needs

$$\langle u, \Delta u \rangle_E = \|\xi\|_2^2. \tag{73}$$

Claim 5.1.

$$\|u\|_{\mathcal{H}_E}^2 = \langle \xi, M\xi \rangle_{l^2}. \tag{74}$$

Proof. (of Claim 2.6). We compute:

$$\begin{aligned} \|u\|_E^2 &= \langle u, u \rangle_E \\ &= \sum_x \sum_y \overline{\xi}_x \xi_y \langle v_x, v_y \rangle \\ &\underset{by\ 69}{=} \sum_x \overline{\xi}_x (M\xi)_x \\ &= \langle \xi, M\xi \rangle_{l^2}, \end{aligned}$$

as claimed. □

The desired conclusion (67) now follows: If $u \in \mathcal{V} \setminus (o)$ is given by (69), then

$$\frac{\langle u, \Delta u\rangle_E}{\|u\|_E^2} = \frac{\|\xi\|_2^2}{\langle \xi, M\xi\rangle}. \tag{75}$$

By taking closure, we obtain the sets on the two sides in (67) □

Corollary 5.1.

If $\xi = (\xi_x) \in \mathcal{F}(G^0 \setminus (o))$, then the representation

$$u = \sum_x \xi_x v_x \tag{76}$$

is unique; in particular, the system $(v_x)_{x \in G^0\setminus(o)}$ is linearly independent.

Proof. Let $u \in \mathcal{V}$ have a representation (76) as a finite summation with $\xi_x \in \mathbb{C}$.
Let $y \in G^0 \setminus (o)$. Then

$$\begin{aligned}\langle \delta_y, u\rangle_E &= \sum_x \xi_x \langle \delta_y, v_x\rangle_E \\ &\underset{\text{by (50)}}{=} \sum_x \xi_x(\delta_y(x) - \delta_y(o)) \\ &= \xi_y.\end{aligned}$$

In particular, if $u = 0$, then $\xi_y = 0$, $\forall y \in G^0 \setminus (o)$. □

Corollary 5.2.

If $F \subset G^0 \setminus (o)$ is a finite subset, then 0 is not in the spectrum of the matrix

$$M_F \equiv (\langle v_x, v_y\rangle_E)_{x,y\in F}. \tag{77}$$

Suppose $o \in \mathrm{spec}(M_F)$ where F is a fixed as in the statement of the Corollary 5.2. Then

$$\exists \xi \in l^2(F \setminus (o))$$

such that

$$(M\xi)_x = \sum_{y\in F} \langle v_x, v_y\rangle_E \xi_y = 0. \tag{78}$$

Setting $u \equiv \sum_{y\in F} \xi_y v_y$ we note that

$$u \in (\{v_x\}_{x\in F})^{\perp}. \tag{79}$$

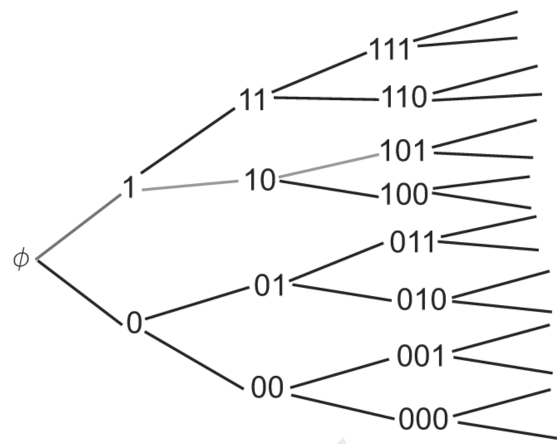

Figure 3. Encoding of vertices.

Claim 5.2.

$$u \in (\{v_x\}_{x\in G^0\setminus(o)})^{\perp}. \tag{80}$$

We need to prove this only if $x \in G^0 \setminus F$.
Combining (74) and (78), we get

$$\begin{aligned}\|u\|_E^2 &= \langle \xi, M\xi\rangle_{l^2} \\ &= \langle \xi, 0\rangle_{l^2} \\ &= 0,\end{aligned}$$

so u = a constant function on G^0, and (80) is satisfied.

6. The energy-inner product

$(G, c) = (\mathrm{tree}\, T, 1)$, $o = \emptyset$, $c \equiv 1$. Explicitly form for v_x, $x \in G^0 \setminus (o)$. Set

$$x = (a_1a_2a_3 \cdots a_n) \in G^0 \setminus (o) \quad a_i \in A = 0, 1. \tag{81}$$

$$\begin{aligned}\gamma(x) = \{&(oa_1), (a_1a_2), (a_2a_3), \\ &\cdots, (a_{n-2}a_{n-1}), (a_{n-1}a_n)\},\end{aligned} \tag{82}$$

where $\gamma(x)$ is a path. Note $\gamma(x) \subset G^1$ = edges in T.

Example 6.1.

$x = 101$ vertex, $\{(\varphi, 1), (1, 10), (10, 101)\} = \gamma(x)$ $\sharp\gamma(x) = 3$.

Theorem 6.1.
Let $(T, 1)$ be as usual, $o = \emptyset$, and let $\mathcal{H}_E$ = the 0 energy span

$$\|f\|_E^2 = \frac{1}{2}\sum_{x}\sum_{\substack{y \\ x\sim y}}(f(x) - f(y))^2, \tag{83}$$

but with edges $(\underline{x}, \underline{xb}) = e$, $x \in G^0$, $b \in A = \{0, 1\}$, $c(e) \equiv 1$. Then the function

$$v_x(y) \equiv \sharp(\gamma(x) \cap \gamma(y)) \tag{84}$$

solves

$$\langle v_x, f\rangle_E = f(x) - f(o), \quad \forall f \in \mathcal{H}_E, \tag{85}$$

$$\Delta v_x = \delta_x - \delta_o, \quad x \in G^0 \setminus (o) \tag{86}$$

and

$$\langle v_x, v_y\rangle_E = \sharp(\gamma(x) \cap \gamma(y)) \quad \forall x, y \in G^0 \setminus (o). \tag{87}$$

Proof. *Proof of (86).* By (84) $x = (a_1 a_2 \cdots a_n) \in G^0 \setminus (o)$. Let x be as in (82). Set $\gamma(x)$ = RHS in (82) $\subseteq G^1$. Neighbors of

$$\begin{aligned} x &\longrightarrow a_1 \cdots a_{n-1} \\ &\longrightarrow x0 \\ &\longrightarrow x1. \end{aligned}$$

If $x = a$, $n = 1$, $\mathrm{Nbh}(x) = \{o, a0, a1\}$.

Cases

n=1 See Fig. 5

$$\begin{aligned} (\Delta v_x)(o) &= 2v_x(o) - v_x(0) - v_x(1) \\ &= 0 - 1 \\ &= \delta_x(o) - \delta_o(o), \end{aligned}$$

$$\begin{aligned} (\Delta v_x)(x) &= 3v_x(x) - v_x(o) - v_x(x0) - v_x(x1) \\ &\underset{use\ (84)}{=} 3 - 0 - 1 - 1 = 1 \\ &= (\delta_x - \delta_o)(x). \end{aligned}$$

Now, let $y \in G^0 \setminus \{o, x\}$. $y = (b_1 b_2 \cdots b_k)$, $b_i \in A = \{0, 1\}$. Suppose $x \subseteq y$

$$\Delta v_x(y) \underset{by\ (84)}{=} 3 - 1 - 1 - 1 = 0.$$

More cases are $\equiv 0$.

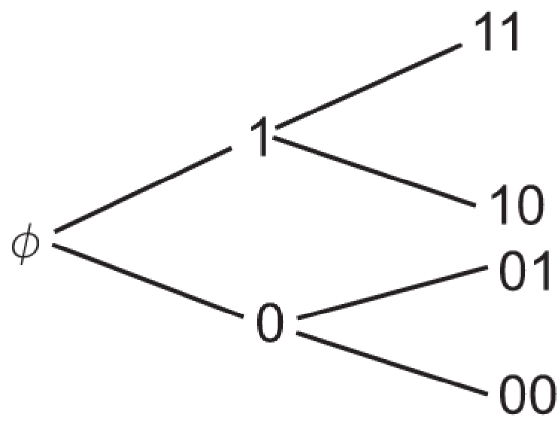

Figure 4. Case 1.

n>1 $x = (a_1 a_2 \cdots a_n)$. A computation yields

$$\Delta_x(o) = 0 - v_x(0) - v_x(1) = 0 - 1 = -1,$$

$$\begin{aligned} \Delta_x(x) &= 3n - (n-1) - 2n = 1 \\ &= (\delta_x - \delta_o)(x), \end{aligned}$$

$$\Delta_x(y) = 0 \quad y \in G^0 \setminus \{o, x\}.$$

Several cases e.g. $y \leq x$, *etc.*

$$\begin{aligned} \Delta_x(y) &= 3v_x(y) - v_x(b_1 \cdots b_{k-1}) - v_x(y0) - v_x(y1) \\ &= 3k - (k-1) - (k+1) - k = 0, \ etc. \end{aligned}$$

Computation of

$$\begin{aligned} \|v_x\|_E^2 &= E(v_x) \\ &= \frac{1}{2}\sum_{s}\sum_{\substack{t \\ s\sim t}} c_{s,t}(v_x(s) - v_x(t))^2 \\ &= \langle v_x, \Delta v_x\rangle_{l^2} \underset{by\ (86)}{=} \langle v_x, \delta_x - \delta_o\rangle_{l^2} \\ &= v_x(x) - v_x(o) = n - 0 \\ &= \sharp(\gamma(x)), \end{aligned}$$

$$\begin{aligned} \langle v_x, v_y\rangle_E &\underset{\text{all finite functions, see (84)}}{=} \langle v_x, \Delta v_y\rangle_{l^2} \\ &\underset{by\ (86)}{=} \langle v_x, \delta_y - \delta_o\rangle_{l^2} \\ &\underset{by\ (84)}{=} v_x(y) - v_x(o) = 0 \\ &\underset{by\ (84)}{=} \sharp(\gamma(x) \cap \gamma(y)). \end{aligned}$$

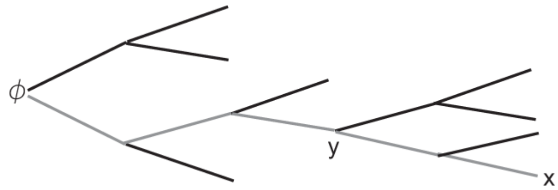

Figure 5. Case 2.

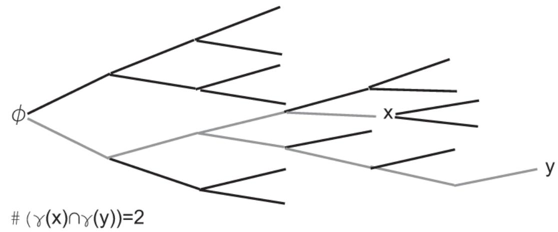

Figure 6. Intersection of two paths.

$\square$

Set $M = (\langle v_x, v_y\rangle_E) = (\sharp(\gamma(x)\cap\gamma(y)))$, $x, y \in G^0 \setminus (o)$. Given

$$\mathrm{spec}_{l^2}(M) = (\mathrm{spec}_{\mathcal{H}_E}(\Delta))^{-1}_{\mathrm{spec}_{\mathcal{H}_E(\Delta)\to\infty}}.$$

From our theorem above Δ (unbounded spectrum), closure$\Delta = \mathcal{V}$, $\mathcal{V} \subset \mathcal{H}_E$.

Corollary 6.1.

$\forall\epsilon \quad \exists F \subset G^0\setminus(o)$ finite, $\exists\lambda \in spec_{l^2}(M_F)$ such that $\lambda < \epsilon$.

Note

$$\begin{aligned} M_F &= (\langle v_x, v_y\rangle_E)_{x,y\in F} \\ &= (\sharp(\gamma(x)\cap\gamma(y)))_{x,y\in F} \end{aligned}$$

and $0 \notin \mathrm{spec}_{l^2}(M_F)$.
Problem: Find a systematic way of selecting F. See Fig. 7.
It is much easier to find M_F with $\mathrm{spec}_{l^2} M_F \to \infty$.

Example 6.2.

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ 1 & n \end{pmatrix}, \tag{88}$$

$$\lambda_n^{\pm} = \frac{n+1 \pm \sqrt{(n+1)^2 - 4(n-1)}}{2}, \quad \text{and}$$

Column Index

		0	1	00	01	10	11	000	001	010	011	100	101	110	111
	0	1	0	1	1	0	0	1	1	1	1	0	0	0	0
	1	0	1	0	0	1	1	0	0	0	0	1	1	1	1
Row Index	00	1	0	2	1	0	0	2	2	1	1	0	0	0	0
	01	1	0	1	2	0	0	1	1	2	2	0	0	0	0
	10	0	1	0	0	2	1	0	0	0	0	2	2	1	1
	11	0	1	0	0	1	2	0	0	0	0	1	1	2	2
	000	1	0	2	1	0	0	3	2	1	1	0	0	0	0
	001	1	0	2	1	0	0	2	3	1	1	0	0	0	0
	010	1	0	1	2	0	0	1	1	3	2	0	0	0	0
	011	1	0	1	2	0	0	1	1	2	3	0	0	0	0
	100	0	1	0	0	2	1	0	0	0	0	3	2	1	1
	101	0	1	0	0	2	1	0	0	0	0	2	3	1	1
	110	0	1	0	0	1	2	0	0	0	0	1	1	3	2
	111	0	1	0	0	1	2	0	0	0	0	1	1	2	3

Figure 7. Google matrix.

$$\begin{aligned} \lambda_n^- &= \frac{n+1-\sqrt{(n+1)^2-4(n-1)}}{2} \\ &= \frac{2(n-1)}{n+1-\sqrt{n^2-2n+2}} \\ &\underset{n\to\infty}{\longrightarrow} 1. \end{aligned}$$

Actually both expand part of $\mathrm{spec}_{l^2} M$ as intervals.

7. Karhunen-Loève

Definition 7.1.

$F \subset G^0 \setminus (o)$ F finite, G^0 infinite, (G, c) fixed. Fix $o \in G^0 \rightsquigarrow \Delta = \Delta_c$, $\mathcal{H}_E$ energy Hilbert space $\langle v_x, f\rangle_E = f(x) - f(o)$, $\forall f \in \mathcal{H}_E$.

Definition 7.2.

$\mathcal{H}_E(F) \equiv \overline{span}_{x\in F}\{v_x\}$

General $(G, c) \to$ Fix point $o \in G^0 \to \Delta_c, \mathcal{H}_E$, G^0 infinite. Fix v_x, for $x \in G^0 \setminus (o)$, determined from Riesz applied to $\mathcal{H}_E$.

$$\langle v_x, f\rangle_E = f(x) - f(o), \quad x \in G^0 \setminus (o), \tag{89}$$

and consider the infinite matrix

$$M = (\langle v_x, v_y\rangle_E), \quad x, y \in G^0 \setminus (o) \tag{90}$$

and its finite $F \times F$ submatrices

$$M_F = (\langle v_x, v_y\rangle_E), \quad x, y \in F, \tag{91}$$

so the matrices are $\infty \times \infty$, or $|F| \times |F|$.

Important formula

Observe

$$\langle v_x, v_y \rangle_E = v_y(x) - v_y(o) \tag{92}$$
$$= v_y(x) = v_x(y); \tag{93}$$

in other words $k_E(x, y) \equiv v_x(y)$ is a reproducing kernel. Since $x, y \in G^0 \setminus (o)$; and $v_x : G^0 \to \mathbb{R}$ (*i.e.*, real valued) convention: $v_x(o) = 0$. Diagonalization motivated by the classical Karhunen-Loève theorem, see [14] and [17].

7.1. Finite-dimensional approximation

Apply the Spectral Theorem to M and M_F. The Hilbert space is $l^2(G^0)$ or $l^2(F) \simeq \mathbb{C}^{|F|}$ with $\langle \xi, \eta \rangle_2 = \sum_x \overline{\xi_x}\eta_x$ as inner product.
For $(M_F, l^2(F))$ the spectrum is always discrete, and for some cases *i.e.*, $(M, l^2(G^0))$ it may not be discrete.
In the discrete case, there exists $M = Mf$ ONB $\xi_1, \xi_2, \ldots \in l^2(G^0)$ or $l^2(F)$ eigenvectors

$$\langle \xi_j, \xi_k \rangle_2 = \sum_x \overline{\xi_j(x)}\xi_k(x) = \delta_{j,k} = \begin{cases} 0 & \text{if } j = k, \\ 1 & \text{if } j \neq k, \end{cases} \tag{94}$$

$\xi = \xi^F \in l^2(F)$ such that

$$M^F \xi_j = \lambda_j \xi_j, \quad \lambda_1 \geq \lambda_2 \geq \cdots > 0,$$
$$\xi_j \in l^2(F), \quad \|\xi_j\|_2 = 1 \quad (\text{in the } F\text{-case}). \tag{95}$$

In the infinite case spec(M) for $l^2(G^0)$ may accumulate both at 0 and at ∞.
Since $M^F_{xy} = \langle v_x, v_y \rangle_{E\Lambda} \in \mathbb{R}$, we may take all $\xi_k : G^0 \to \mathbb{R}$ real valued. Fix $F \subset G^0 \setminus (0)$: $\xi^F_k \in l^2(F)$. Set

$$w^F_k(\cdot) = \frac{1}{\lambda_k} \sum_{x \in F} \xi^F_k(x) v_x(\cdot)$$

i.e.,

$$w^F_k(z) = \frac{1}{\lambda_k} \sum_{x \in F} \xi^F_k(x) v_x(z), \quad \forall z \in G^0. \tag{96}$$

Lemma 7.1.
If F is fixed then $\xi^F_k \in l^2(F)$ is an ONB. Set

$$M_F \xi^F_k = \lambda^F_k \xi^F_k, \tag{97}$$

then

$$w^F_k : G^0 \to \mathbb{R}, \quad w^F_k \in \mathcal{H}_E$$

is an extension of $\xi^F_k : F \to \mathbb{R}$ from F to G^0.

Proof. By (96) if $z \in F$:

$$\begin{aligned} w^F_k(z) &= \frac{1}{\lambda_k} \sum_{x \in F} v_x(z) \xi^F_k(x) \\ &\underset{\text{by}(4.4)}{=} \frac{1}{\lambda_k} \sum_{x \in F} M_{z,x} \xi^F_k(x) \\ &\underset{\text{by (97)}}{=} \frac{1}{\lambda_k} (M_F \xi^F_k)_z \\ &= \frac{\lambda_k}{\lambda_k} \xi^F_k(z) \\ &= \xi^F_k(z). \end{aligned}$$

□

Lemma 7.2.
Fix $F \subset G^0 \setminus (0)$ finite, and let

$$w^F_k(\cdot) = \frac{1}{\lambda_k} \sum_{x \in F} \xi^F_k(x) v_x(\cdot), \quad k \in (1, 2, \cdots, |F|). \tag{98}$$

as in Lemma 7.1. Then $\{w^F_k\}_k$ is an orthonormal system in H_E (thus in each of the Hilbert spaces) i.e., with the inner product

$$\langle u, v \rangle_E \equiv \frac{1}{2} \sum_{\text{all } xy} \sum_{x \sim y} c_{xy} (\overline{u(x)} - \overline{u(y)})(\overline{v(x)} - \overline{v(y)}). \tag{99}$$

We have

$$\langle w^F_j, w^F_k \rangle_E = \frac{1}{\lambda_k} \delta_{j,k} = \begin{cases} \frac{1}{\lambda_k} & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \tag{100}$$

Proof. We have:

$$\begin{aligned} \langle w^F_j, w^F_k \rangle_E &\underset{\text{by (98)}}{=} \frac{1}{\lambda_j \lambda_k} \sum \sum_{xy \in F} \xi^F_j(x) \xi^F_k(y) \langle v_x, v_y \rangle_E \\ &\underset{\text{by (91)}}{=} \frac{1}{\lambda_j \lambda_k} \sum_{x \in F} \xi^F_j(x) \sum_{y \in F} M^F_{xy} \xi^F_k(y) \\ &= \frac{1}{\lambda_j \lambda_k} \langle \xi^F_j, M^F \xi^F_k \rangle_2 \\ &= \frac{1}{\lambda_j} \langle \xi^F_j, \xi^F_k \rangle_2 \\ &\underset{\text{by (94)}}{=} \frac{1}{\lambda_j} \delta_{j,k} = \begin{cases} \frac{1}{\lambda_j} & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases} \end{aligned}$$

□

Set $u^F_j = \sqrt{\lambda_j} w^F_j$; then

$$\langle u^F_j, u^F_k \rangle_E = \delta_{j,k}, \quad j, k \in \{1, 2, \cdots, |F|\}.$$

7.2. Normalization

The following different normalization $u_j^F = \sqrt{\lambda_j} w_j^F$ satisfies

$$\|u_j^F\|_{H_E} = 1, \tag{101}$$

so

$$u_j^F(\cdot) = \frac{1}{\lambda_j} \sum_{x \in F} \xi_j(x) v_x(\cdot). \tag{102}$$

Note that the

$$u_j^F|_F = \sqrt{\lambda_j} \xi_j(\cdot) \quad \text{on } F. \tag{103}$$

7.3. Projection valued measures

Set

$$P^F(\lambda_j) \equiv |u_j^F >< u_j^F|; \tag{104}$$

Dirac notation for rank-one projection, so a projection in H_E on the one-dimensional subspace $\mathbb{C}u_j^F$. Then $P^F(\cdot)$ is an orthonormal projection system, and it has a limit as $F \to \infty$ which is a global spectral measure.
We claim that

$$s_\Delta(u_j^F) = \langle u_j^F, \Delta u_j^F \rangle \in spec_{H_E}(\Delta v). \tag{105}$$

Lemma 7.3.
(Spectral Reprocity)

$$s_\Delta(u_j^F) = \frac{1}{\sqrt{\lambda_j}} \left(1 + \left| \sum_{x \in F} \xi_j(x) \right|^2 \right). \tag{106}$$

Proof.

$$\begin{aligned} s_\Delta(u_j^F) &\underset{by\ (102)}{=} \frac{1}{\sqrt{\lambda_j}} \left\langle \left(\sum_{x \in F} \xi_j(x) v_x \right), \Delta \left(\sum_{y \in F} \xi_j(y) v_y \right) \right\rangle_E \\ &= \frac{1}{\sqrt{\lambda_j}} \sum_{x \in F} \sum_{y \in F} \xi_j(x) \xi_j(y) \langle v_x, \Delta v_y \rangle_E \\ &= \frac{1}{\sqrt{\lambda_j}} \sum_{x \in F} \sum_{y \in F} \xi_j(x) \xi_j(y) (\delta_x(y) + 1) \\ &= \frac{1}{\sqrt{\lambda_j}} \left(\|\xi_j\|_2^2 + \left| \sum_{x \in F} \xi_j(x) \right|^2 \right) \\ &\underset{by\ (101)}{=} \frac{1}{\sqrt{\lambda_j}} \left(1 + \left| \sum_{x \in F} \xi_j(x) \right|^2 \right). \end{aligned}$$

□

Example 7.1.

$$M^F = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad sp = \{1, 3\},$$

same spectrum, but different M^F.

$$\xi_{\lambda=1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_{\lambda=3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \langle \xi_1 \rangle = \langle \xi_3 \rangle = 1.$$

Set $R_F(\lambda) \equiv \frac{1}{\lambda}(1 + |\langle \xi_\lambda^F \rangle|^2)$. Then $\langle u_\lambda, \Delta u_\lambda \rangle = R_F(\lambda)$; see Lemma 7.3.
In the examples:

$$M^F = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{cases} R_F(1) = 1, \\ R_F(3) = \frac{1}{3}(1+2) = 1, \end{cases}$$

smaller for M^F off-diagonal.

$$M^F = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{cases} R_F(1) = \frac{1}{1}(1+1^2) = 2, \\ R_F(3) = \frac{1}{3}(1+1^2) = \frac{2}{3}, \end{cases}$$

$$M^F = \begin{pmatrix} 3 & 3 \\ 3 & 7 \end{pmatrix}, \quad \lambda_\pm = 5 \pm \sqrt{13},$$

$$M^F = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}, \quad \lambda_\pm = \frac{7 \pm \sqrt{5}}{2},$$

$$\lambda = 5 \pm \sqrt{13} \Rightarrow R(\lambda) = \frac{1}{\lambda} + \frac{\lambda}{1 + (\frac{2-\sqrt{13}}{3})^2} < \frac{1}{\lambda} + \lambda,$$

$$M^F = \begin{pmatrix} 1 & 1 \\ 1 & m \end{pmatrix}, \quad m \to \infty,$$

$$\lambda_\pm = \frac{m + 1 \pm \sqrt{(m+1)^2 - 4(m-1)}}{2}.$$

In both cases, we have:

$$R_F(\lambda) = \frac{1}{\lambda} + \frac{\lambda}{1 + (\lambda - 1)^2},$$

$$R_F(\lambda_-) = \frac{\lambda_+}{m-1} + \frac{\lambda_-}{1 + (\lambda_- - 1)^2} \sim 4 \quad \text{as } m \to \infty.$$

We now illustrate by an example that points in the spectrum can go into ∞:

$$s_\Delta(u) = \frac{\langle u, \Delta u \rangle}{\|u\|_E^2} \to \infty.$$

If $\lambda \in spec_{l^2}(M^F)$ set $u_\lambda = \frac{1}{\sqrt{\lambda}} \sum_{x \in F} \xi_\lambda(x) v_x(\cdot)$ $M\xi_\lambda = \lambda \xi_\lambda$, $\|\xi_\lambda\|_2 = 1 \Rightarrow \|u_\lambda\|_E = 1$ so $s_\Delta(u) = \langle u, \Delta u \rangle = \frac{1}{\lambda}(1 + \|P_\lambda e\|_2^2)$, $e = e_F = \chi_F(\cdot)$, $P_\lambda e = \langle \xi_\lambda, e \rangle_2 \xi_\lambda$.

	0	1	10	11	100	101	110	111
0	1	0	0	0	0	0	0	0
1	0	1	1	1	1	1	1	1
10	0	1	2	1	2	2	1	1
11	0	1	1	2	1	1	2	2
100	0	1	2	1	3	2	1	1
101	0	1	2	1	2	3	1	1
110	0	1	1	2	1	1	3	2
111	0	1	1	2	1	1	2	3

Figure 8. Table of distances.

Theorem 7.1.

The truncated operators $P_{\mathcal{H}_E(F)}\Delta_{\mathcal{D}_E}P_{\mathcal{H}_E(F)}$ has spectral growth $\simeq \mathcal{O}(\#F)$; so Δ_E is unbounded in $\mathcal{H}_E$.

Proof. The idea is to perform a diagonalization of an infinite matrix $(M_{x,y})$ $x,y \in G^0\setminus(o)$; a method inspired by Karhunen-Loève [14, 17]. Here $F \subset G^0\setminus(o)$ is fixed and finite. The following computations refer to F: (ξ_k) is an ONB in $l^2(F)$ satisfying (107) below; set $w_k = \frac{1}{\lambda_k}\sum_{x\in F}\xi_k(x)v_x$, and $v_k = \sqrt{\lambda_k}w_k = \frac{1}{\sqrt{\lambda_k}}\sum_{x\in F}\xi_k(x)v_x$. Then

$$M^F\xi_k = \lambda_k\xi_k, \quad \text{and} \quad \langle\xi_j,\xi_k\rangle_{l^2(F)} = \delta_{j,k}. \tag{107}$$

We may now compute the matrices:

$$\begin{aligned}\langle u_j,\Delta u_k\rangle_E &= \frac{1}{\sqrt{\lambda_j\lambda_k}}\sum\sum_{F\times F}\xi_j(x)\xi_k(x)\langle v_x,\Delta v_y\rangle_E\\ &= \frac{1}{\sqrt{\lambda_j\lambda_k}}\sum\sum_{F\times F}\xi_j(x)\xi_k(x)(\delta_x(y)+1).\end{aligned}$$

Set $\sum_{x\in F}\xi_j(x) = \langle\xi_j,e\rangle_2 = \langle\xi_j\rangle$ where $e = e^F = \chi_F$. Then the matrix entries are: Off-Diagonal:

$$\langle u_j,\Delta u_k\rangle_E = \frac{1}{\sqrt{\lambda_j\lambda_k}}(\delta_{j,k}+\langle\xi_j\rangle\langle\xi_k\rangle);$$

and Diagonal:

$$\langle u_j,\Delta u_j\rangle_E = \frac{1}{\lambda_j}(1+\langle\xi_j\rangle^2).$$

We further used the following identity:

$$\begin{aligned}\sum\sum_{F\times F}\delta_x(y)\xi_j(x)\xi_k(y) &= \langle\xi_j,\xi_k\rangle_{l^2}(F)\\ &= \delta_{j,k} \quad \text{by (107)}.\end{aligned}$$

This may be summarized in the following matrix form:

$$\begin{pmatrix}\frac{1}{\lambda_1}(1+\langle\xi_1\rangle^2) & \frac{\langle\xi_1\rangle\langle\xi_2\rangle}{\sqrt{\lambda_1\lambda_2}} & \frac{\langle\xi_1\rangle\langle\xi_3\rangle}{\sqrt{\lambda_1\lambda_3}} & \cdots\\ \frac{\langle\xi_1\rangle\langle\xi_2\rangle}{\sqrt{\lambda_1\lambda_2}} & \frac{1}{\lambda_2}(1+\langle\xi_2\rangle^2) & \frac{\langle\xi_2\rangle\langle\xi_3\rangle}{\sqrt{\lambda_2\lambda_3}} & \cdots\\ \frac{\langle\xi_1\rangle\langle\xi_3\rangle}{\sqrt{\lambda_1\lambda_3}} & \frac{\langle\xi_2\rangle\langle\xi_3\rangle}{\sqrt{\lambda_2\lambda_3}} & \frac{1}{\lambda_3}(1+\langle\xi_3\rangle^2) & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}.$$

If for some $\delta \in \mathbb{R}^+$, $\lambda j \geq \delta$, *i.e.*, bounded from below, then the operator

$$\begin{pmatrix}\frac{1}{\lambda_1} & 0 & 0 & 0 & \cdots & 0\\ 0 & \frac{1}{\lambda_2} & 0 & 0 & \cdots & 0\\ 0 & 0 & \frac{1}{\lambda_3} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & 0\end{pmatrix}$$

is bounded. So

$$\left(\frac{1}{\sqrt{\lambda_j\lambda_k}}\langle\xi_j\rangle\langle\xi_k\rangle\right) \tag{108}$$

must be unbounded, *i.e.*, $\|\cdot\|_{l^2(F)\to l^2(F)} \to \infty$. But (108) is a rank-one operator;

$$\begin{aligned}&|\rho><\rho|, \quad \rho = \rho^F; \quad F\subset G^0\setminus(o) \text{ is fixed,}\\ &\qquad\qquad \text{where } \rho = (e_j) \in l^2(1,2,\cdots,\#F),\end{aligned}$$

i.e., $\rho = \rho^F$ and $\rho_j^F = \frac{\langle\xi_j^F\rangle}{\sqrt{\lambda_j}}$, $\lambda_j = \lambda_j^F$.

Now,

$$\|\rho^F\|^2_{l^2(1,\cdots,\#F)} = \sum_{j=1}^{\#F}\frac{\langle\xi_j^F\rangle^2}{\lambda_j}.$$

So in conclusion

$$\lim_{F\to\infty}\sum_{j=1}^{\#F}\frac{\langle\xi_j^F\rangle^2}{\lambda_j(F)} = \infty.$$

Pick $\delta \in \mathbb{R}_+$ and assume $\lambda_j^F \geq \delta$. Then we need

$$\lim_{F\to\infty}\sum_{j=1}^{\#F}\langle\xi_j^F\rangle^2 = \infty.$$

We have $\xi_\lambda^F(j) = \xi_j$, $M^F\xi_\lambda^F = \lambda_j^F\xi_\lambda^F$, $\|\xi_\lambda\|_2 = 1$, $\langle\xi_\lambda^F\rangle = \sum_{x\in F}\xi_\lambda^F(x)$, and $\sum_\lambda\langle\xi_\lambda^F\rangle^2 = \#F$; so indeed

$$\lim_{F\to\infty}\sum_\lambda\langle\xi_\lambda^F\rangle^2 = \lim_F \#F = \infty.$$

Conclusion: $spec_{\mathcal{H}_E(N)}(\Delta_{\mathcal{D}_E}) \sim (\#F) \to \infty$.

□

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